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Valuing Firms Under Default Risk and Bankruptcy Costs: A WACC-Based Approach

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ABSTRACT

A general methodology for valuing leveraged firms under default risk and bankruptcy costs is discussed. The analysis concerns with company valuation based on Discounted Cash Flow (DCF) techniques and on the Weighted Average Cost of Capital (WACC) method. Within this framework, a WACC based model is presented and a simple WACC formula under constant debt ratio is derived. Depending on one-period default probabilities, WACC rates are not necessarily constant over time, nevertheless a closed form expression is obtained. The WACC formula derived in this paper accounts for tax shields and bankruptcy costs through the unconditional probability of default and, in this way, WACC rates include credit spreads of corporate debt. The optimal financial structure arises in a quite natural way within the context of the proposed model as a consequence of the trade-off between tax shields benefits and bankruptcy costs. To illustrate the flexibility of the model, a numerical analysis is proposed.

JEL Classifications: *G31, G32, G33*

Keywords: *risky debt; bankruptcy costs; tax shield; present value; WACC*

I. INTRODUCTION

The literature on default risk modeling followed two main directions characterized by well defined and distinct pricing schemes. The first direction is based on contingent claim valuation techniques and was pioneered by Merton (1974). In his seminal paper the default event is triggered endogenously by the inability of the firm to meet its debt obligations. The valuation of risky debt is then performed using contingent claim analysis assuming that the value of the firm activities (the underlying security) follows a diffusion process. Such an approach was then extended by Leland (1994) to include taxes and bankruptcy costs. Under the hypothesis of infinite life debt with a constant face value, Leland obtained closed form solutions for (time independent) risky debt, credit spreads and for the optimal capital structure. The effect of the debt maturity and of the amount of debt issued by the firm was then examined by Leland and Toft (1996). Within the same pricing scheme, Duffie and Singleton (1999) proposed a different methodology for modeling term structures of defaultable bonds in which the default event is regarded as exogenous and it is modeled as, for example, the first jump of a Poisson process. In this reduced form approach, the valuation of defaultable bonds is performed using contingent claim analysis under well defined recovery rate conditions. This approach revealed flexible enough to be used in practical applications (see, e.g., Mari and Renó, 2005).

Alternative to contingent claim analysis, the Discounted Cash Flow (DCF) method is a widely applied pricing scheme for company valuation purposes, also in presence of defaultable debt. It is based on the very general idea that the valuation process can be performed by discounting the future expected cash flows at a rate, namely a risk-adjusted discount rate, which accounts for the risk of cash flows (Modigliani and Miller, 1963). Due to its mathematical tractability, the risk-adjusted discount rate approach is a widely applied methodology, especially in the context of the Adjusted Present Value (APV) method (Myers, 1974) or using the Weighted Average Cost of Capital (WACC) method (Miles and Ezzel, 1980). Although the literature on optimal capital structure accounted for credit risk and bankruptcy costs using contingent claim analysis since the Nineties, nevertheless these features have not been included in a satisfactory way within the context of the DCF approach (Koziol, 2014). This paper would be a further and deep insight into this line of research.

Two are the main contributions of this study to the literature. First, the paper provides a general methodology based on the DCF pricing technique for valuing leveraged firms under default risk and bankruptcy costs. It will be shown that such an approach is valid under very general conditions on the default event modeling and without any constraints about the debt policy of the firm. Within this context, a generalization of Proposition II of the Modigliani-Miller theorem (Modigliani and Miller, 1958) is obtained. Moreover, WACC rates accounting for defaultable debt and bankruptcy costs are derived. Such a WACC formula extends the classical Modigliani-Miller formula (1963), the Harrison-Pringle formula and the formula derived by Farber, Gillet and Szafarz (2006) in the case in which bankruptcy costs are included into the analysis. Moreover, it accounts for credit spreads of corporate debt in a natural way through unconditional single-period default probabilities. The second contribution of this study is to provide, within the context of the general methodology proposed in this paper, a simple and mathematically tractable model to value firms under default risk when bankruptcy costs are included. The default, treated as an exogenous event, can occur at any time during the lifetime of the firm and the valuation is performed using a DCF technique based on the WACC method under the hypothesis of constant debt ratio policy[1]. Although the default event is assumed to be exogenous, the default probability depends on the debt ratio and, as a consequence, it depends on the credit spread of the firm's debt. A simple WACC formula that takes into account in a proper way the risk of distress cost is derived. We will show that such a formula is a powerful tool of analysis for valuing firms under default risk and bankruptcy costs and provides a more rigorous representation of WACC rates with respect the WACC formula proposed in the literature by Koziol (2014).

The WACC method is a powerful tool of analysis because it allows to determine the market value of a firm by discounting unlevered cash flows at WACC rates. Tax shields effects as well as distress costs effects are explicitly included in the WACC formula. The cost of debt is an unnecessary firm valuation parameter. This fact is well known when interests tax saving is not allowed and distress costs are negligible, as a consequence of the Modigliani-Miller theorem. If tax shields are allowed and under not negligible bankruptcy costs, the market value of a leveraged firm can be expressed as the sum of the market value of an otherwise identical but unlevered firm plus the present value of tax shields minus the present value of bankruptcy costs. Although present values of tax shields and bankruptcy costs depend on credit spreads of corporate debt, nevertheless WACC rates are explicitly independent on the cost of corporate debt. The WACC formula we derive in this paper accounts for tax shields and bankruptcy costs through the unconditional probability of default and in this way WACC rates include credit spreads of corporate debt. To obtain a consistent WACC formula, we must take into account present values of tax shields and bankruptcy costs in a proper way. In particular, WACC rates must include appropriate discount rates for tax shields and distress costs valuation. In this paper we show how to incorporate such rates into a simple WACC formula which can be used in practical applications. To do this two main problems, then, arise: the first one is to find the discount rate for tax saving from defaultable debt which properly accounts for the tax shields risk profile; the second is to determine the appropriate discount rate for distress costs.

A very reach debate exists on the first topic. From the seminal papers by Miles and Ezzell (1980, 1985), the literature on this argument has grown exponentially. To quote some milestones, it is important to mention Harris and Pringle (1985), Ruback (2002), Cooper and Nyborg (2006, 2008), Molnaàr and Nyborg (2013). One of the main result was that under constant debt ratio unlevered rates offer a good approximation of tax shields discount rates. Although Menichini (2016) finds that for a representative U.S. firm the value of the tax shields represents less than 5% of the firm value, the theoretical research on the appropriate discount rate for tax shields valuation is still rather rich (see, e.g., Krause and Lahmann, 2016). Textbooks as Brealey and Myers (2003) or Berk and De Marzo (2014) offer a comprehensive description of the problem. The references contained in the above quoted papers and textbooks give an exhaustive view of the argument.

Bankruptcy costs are a very special class of costs that arise when the firm does poorly. During a default process a firm exhibits direct and indirect costs that can assume a considerable value of the firm (Altman, 1984; Opler and Titman, 1994). Andrade and Kaplan (1998) estimate losses on the order of 10% to 23% of the predistress firm value. In a more recent analysis, Glover (2016) confirms that the average costs among defaulted firms is about 25% of the predistress firm value. The discount rate for the distress costs will depend on the firm market risk. Since such costs are high when the firm does poorly, the beta of distress costs will have an opposite sign with respect to that of the firm (Almeida and Philippon, 2007; Korteweg, 2010). If the firm has a positive beta, distress costs have a negative beta and the relative discount rate is lower than the risk-free rate. The possibility to use a zero discount rate has been discussed in literature (Warner, 1977). However, due to the magnitude of the distress cost beta, the discount rate may assume negative values[2]. Koziol (2014) provided a simple correction to the standard WACC formula to include default risk and bankruptcy costs under a constant debt ratio policy. Such formula is valid under constant default probabilities and under the rather strong assumption that the unlevered cost of capital is equal to the so called pre-tax WACC. As it will be shown in the paper, such an assumption is in general a poor approximation because it requires that both, the tax shields cash flow and the distress costs cash flow can be discounted at the unlevered cost of capital. For discounting distress cost cash flow this is a rather strong assumption. However, in the case of low debt ratio and low distress costs, such an approximation works quite well and it can be used in practical applications. In the model we propose, discount rates for distress costs are determined by taking into account in a careful and proper way the risk of the distress costs cash-flow. It will be shown that the Koziol formula can be obtained as a limiting case of our WACC formula for low debt ratios and low distress costs levels, under the further hypothesis that one-period default probabilities are constant over time.

Based on a set of assumptions regarding the structure of the tax shield and of the structure of the distress costs which are standard in the literature (Leland, 1994; Koziol, 2014), the model we propose is mathematically tractable and can be used in practical applications to value leveraged firms and determine the optimal financial structure. To show the flexibility of the model a numerical analysis is proposed in the paper. The trade-off between the tax shields contribution to the firm value and the distress costs contribution is shown as a function of the debt ratio. In this context, the optimal financial structure arises in a quite natural way as the level of the debt ratio that maximizes the firm value. Moreover, a numerical test on the validity of the main hypotheses of the model is provided.

The paper is organized as follows. Section II presents a general methodology for valuing leveraged firms under default risk and bankruptcy costs (the derivation of the main formulas presented in this Section is provided in Appendix A). The model is discussed in Section III. The set of the model assumptions is discussed and the WACC valuation formula is derived (Appendix B contains the derivation of the main formulas presented in Section III). The optimal capital structure of the firm is discussed in Section IV within the context of a numerical simulation of the model. Some remarks conclude the paper.

II. VALUING FIRMS UNDER DEFAULT RISK AND BANKRUPTCY COSTS: A GENERAL APPROACH

In this Section we propose a general methodology to value leveraged firms under default risky debt and bankruptcy costs. Two are the main results that will be presented and discussed. First, we derive a generalization of the Proposition II of the Modigliani-Miller theorem when tax shields effects of defaultable debt and bankruptcy costs are taken into account. Second, we extend the definition of WACC rates in the same context. General relationships between WACC and unlevered rates and between WACC and levered rates will be then derived. The model we propose in this paper is developed within this general valuation scheme and it will be presented in the next Section.

Let us denote by $\{F_t^X\}_{t=1}^m$ a collection of measurable random variables with respect a given filtration of a given probability space (Duffie, 1998), describing a stream of stochastic payments, namely a cash flow, and by V_t^X the present value at time t (t=1,2,…,m) of the cash flow. The superscript X stands for U (unlevered), S (levered), D (debt), TS (tax shields) and DC (default costs). By definition, the levered cash flow (or flow to equity), F_t^S, can be expressed as:

$$F_t^S = F_t^U - F_t^D + F_t^{TS} - F_t^{DC} \tag{1}$$

obtained adding to the unlevered cash flow, F_t^U, i.e. the cash flow accounting for the activities of the firm, the tax shield contribution, F_t^{TS} and subtracting the debt repayment, F_t^D, and the distress costs, F_t^{DC}. From Equation (1) and using a no-arbitrage argument, we can express the value at time t of a leveraged firm, V_t, in the following way:

$$V_t \equiv V_t^S + V_t^D = V_t^U + V_t^{TS} - V_t^{DC} \tag{2}$$

The firm value, i.e. the sum of its equity, V_t^S, and of the outstanding debt, V_t^D, can be obtained adding to the value of the unlevered firm, V_t^U, the present value of tax shields, V_t^{TS}, and subtracting the present value of bankruptcy costs, V_t^{DC}. Expected present values are computed discounting expected future cash flows at deterministic risk adjusted discount rates (Mari and Marra, 2017). If we denote by k_t^X the cost of capital (i.e. the discount rate) accounting for the risk of the cash flow F_t^X, the following recursive relation holds:

$$E_0[V_t^X] = \frac{E_0[F_{t+1}^X + V_{t+1}^X]}{1+k_t^X} \tag{3}$$

where E_0 is the conditional expectation operator under the information available at the present time (or time 0). The iterative application of Equation (3) is straightforward and allows to express the expected value at time t of a risky cash flow as

$$E_0[V_t^X] = \sum_{j=t+1}^{m} \frac{E_0[F_j^X]}{\prod_{i=t}^{j-1}(1+k_i^X)} \tag{4}$$

A. Generalizing Proposition II of the Modigliani-Miller Theorem

In force of Equation (2), equity rates, k_t^S, are related to unlevered rates, k_t^U and debt rates, k_t^D, by the linear combination:

$$k^S{}_t = k^U{}_t + (k^U{}_t - k^D{}_t)\frac{E_0[V^D{}_t]}{E_0[V^S{}_t]} - \left(k^U{}_t - k^{TS}{}_t\right)\frac{E_0[V^{TS}{}_t]}{E_0[V^S{}_t]} + (k^U{}_t - k^{DC}{}_t)\frac{E_0[V^{DC}_t]}{E_0[V^S_t]} \qquad (5)$$

in which tax shields rates, k^{TS}_t, and default costs rates, k^{DC}_t, explicitly appear. Appendix A contains a detailed proof of the above result.

Equation (5) accounts for the risk of the levered cash flow due to the presence of risky debt, tax shields and distress costs. It provides a generalization of Proposition II of the Modigliani-Miller theorem. The entity of the equity risk depends on several variables. Among the others, the ratio between the expected value of the outstanding debt and the expected equity value is the most important one. Since $k^U_t \geq k^D{}_t$, equity rates show a non decreasing behavior as this ratio rises. Also, the ratio between the expected value of tax shields and the expected equity value plays an important role. Depending on the sign of the difference between debt and tax shields rates, the third term in the r.h.s. of Equation (5) may increase or decrease equity rates. The last term accounts for default costs and, as it will be shown in the next Section, its contribution may be very relevant. Discount rates for the bankruptcy costs, k^{DC}_t, depend on the firm's market risk. Since distress costs are high when the firm does poorly, the beta of distress costs will have an opposite sign with respect to that of the firm. If the firm has a positive beta, distress costs have a negative beta and the relative discount rate is lower than the risk-free rate. Due to the magnitude of the distress cost beta, the discount rate may assume negative values. For positive betas, this last term has a positive value and accounts for the increase of the equity risk due to bankruptcy costs.

B. The Weighted Average Cost of Capital (WACC)

The WACC method of firm valuation is a powerful tool of analysis because it allows to determine the market value of a firm by discounting unlevered cash flows at WACC rates. Under default risk and bankruptcy costs, WACC rates, k^W_t, can be introduced in the following way. By definition, the expected value of a leveraged firm can be determined discounting the expected unlevered cash flow at WACC rates. They can be, therefore, defined by the following recursive relation:

$$E_0[V_t] = \frac{E_0[F^U_{t+1} + V_{t+1}]}{1 + k^W_t} \qquad (6)$$

WACC rates are related to unlevered rates by the linear combination:

$$k^W_t = k^U{}_t - \left(k^U{}_t - k^{TS}{}_t\right)\frac{E_0[V^{TS}{}_t]}{E_0[V_t]} + \left(k^U{}_t - k^{DC}{}_t\right)\frac{E_0[V^{DC}{}_t]}{E_0[V_t]} - \frac{E_0[F^{TS}{}_{t+1}]}{E_0[V_t]} + \frac{E_0[F^{DC}{}_{t+1}]}{E_0[V_t]} \qquad (7)$$

in which tax shields and distress costs discount rates explicitly appear. Appendix A contains a detailed proof of Equation (7).

Tax shields effects as well as distress costs effects are explicitly included in the WACC formula. Credit risk spread is also included in the WACC formula through the unconditional default probabilities which affects the third and fifth term in the r.h.s. of equation (7).

We notice that WACC formula does not depend on the cost of the corporate debt.

It is the main valuation tool of our model. Due to the presence of tax shields and distress costs discount rates, it shows a rather complicated structure. In the next Section we will show that, on the basis of some specific assumptions, tax shields and distress costs rates can be expressed in terms of unlevered rates and that Equation (7) can be greatly simplified.

WACC rates can be also expressed in terms of equity and debt rates, thus getting:

$$k^{W}_{t} = \frac{E_0[V^{TS}_{t}]}{E_0[V_t]} k^{S}_{t} + \frac{E_0[V^{D}_{t}]}{E_0[V_t]} k^{D}_{t} - \frac{E_0[F^{TS}_{t+1}]}{E_0[V_t]} + \frac{E_0[F^{DC}_{t+1}]}{E_0[V_t]} \quad (8)$$

Appendix A contains a detailed proof of Equation (8).

Iterative applications Equation (6) provide the expected value of a leveraged firm at time t in term of the present value of the expected future unlevered cash flow,

$$E_0[V^{X}_{t}] = \sum_{j=t+1}^{m} \frac{E_0[F^{U}_{j}]}{\prod_{i=t}^{j-1}(1+k^{W}_{i})} \quad (9)$$

We conclude this Section noticing that in presence of defaultable debt and considering distress costs, the (improperly called) pre-tax WACC

$$k^{W^*}_{t} \equiv \frac{E_0[V^{S}_{t}]}{E_0[V_t]} k^{S}_{t} + \frac{E_0[V^{DC}_{t}]}{E_0[V_t]} k^{D}_{t} \quad (10)$$

doesn't coincides with the unlevered rate k^{U}_{t}. In fact, by comparing Equation (7) and Equation (8), the following relation holds:

$$k^{W^*}_{t} = k^{U}_{t} - (k^{U}_{t} - k^{TS}_{t}) \frac{E_0[V^{TS}_{t}]}{E_0[V_t]} + (k^{U}_{t} - k^{DC}_{t}) \frac{E_0[V^{DC}_{t}]}{E_0[V_t]} \quad (11)$$

which clearly shows that the pre-tax WACC differs from the unlevered cost of capital. They coincide when both, the tax shields cash flow and the distress costs cash flow can be discounted at the unlevered cost of capital. As we will shown in the next Section, this is a rather strong assumption, at least for discounting distress cost cash flow.

III. THE MODEL

In the previous Section, a general approach for valuing leveraged companies under defaultable debt and bankruptcy costs has been discussed. Such a valuation scheme is valid without any constraints about the debt policy of the firm and under general conditions on the default event triggering. Starting from these results, we propose a simple model to value leveraged firms with positive beta. The model, based on the set of technical assumptions described below, is mathematically tractable and provides a simple WACC formula to be used in practical applications.

As a basic hypothesis, we consider leveraged firms with a well specified debt ratio target which is constant over time. We state, therefore, the following assumption:

Assumption 0:

$$\frac{V^{D}_{t}}{V_t} = \text{const} \equiv \frac{D}{V} \quad (11)$$

The empirical literature on capital structure documented the existence of a target leverage and investigated the way in which firms adjust the leverage toward their target debt ratios (Byoun, 2008; Dang and Garret, 2015).

The next assumption concerns discount rates for the tax shields cash flow. Since the debt ratio must be constant over time, we assume that the tax shields cash flow can be discounted at unlevered rates:

Assumption 1: $$k_t^{TS} = k^U{}_t \tag{12}$$

This assumption is well documented in literature (Miles and Ezzel, 1980; Cooper and Nyborg, 2006; Oded and Michel, 2007; Dempsey, 2013), and greatly simplifies the WACC formula (7) because the second term in the r.h.s. of Equation (7) vanishes. Assumption 1 is a good approximation for riskless debt, but it works well also in the case of default risky debt, specially if the one-period default probabilities are not too high. In such a case, the risk profile of the tax shield is not too much different from the risk profile of the unlevered cash flow. Moreover, in Equation (7) the difference between the unlevered rate and the tax shields discount rate must be multiplied by the ratio between the present value of tax shields and the firm value. Since such a ratio is of the order of some percent under a constant debt ratio, specially when default risk is considered (Graham, 2001; Couch *et al.*, 2012), the impact of the second term in the r.h.s. of Equation (7) is negligible. The main contribution to the present value of tax shields comes from the ratio$E_0\left[F^{TS}_{t+1}\right]/E_0[V_t]$.

The third term in the r.h.s. of Equation (7) also gives an important contribution to the WACC rates. As we will show in the numerical analysis proposed in Section 4, such contribution mainly depends on the survivorship probability magnitude (that depends also on the debt ratio) and on the entity of distress costs. The numerical analysis provides also a quantification of the contribution of this term as a function of the debt ratio for different levels of distress costs.

As mentioned in the previous Section, if the firm has a positive beta, distress costs have a negative beta and the relative discount rate must be lower than the risk-free rate. Due to the magnitude of the distress cost beta, the discount rate may assume negative values. Taking into account the above considerations, it is evident that unlevered rates are not adequate to discount the distress costs cash flow. The second assumption concerns, therefore, the definition of a *no-growth* condition to keep distress costs discount rates much lower than unlevered rates, allowing them to become negative. To do this, let us express the third term in the r.h.s. of Equation (7) in the following form:

$$(k_t^U - k_t^{DC})\frac{E_0[V^S{}_t]}{E_0[V_t]}k^S{}_t \equiv k_t^U f(t) \tag{13}$$

or equivalently

$$k_t^{DC} = \; k_t^U(1 - f(t)\frac{E_0[V_t]}{E_0[V_t^{DC}]}) \tag{14}$$

If we assume that

$$f(t) = \frac{E_0[V_t^{DC}]}{E_0[V_t]} \quad (15)$$

we obtain $k_t^{DC} = 0$. Although the zero-discount rates condition has been discussed in literature as an interesting approximation for distress costs discount rates (Warner, 1977), we will modify it by allowing for negative rates. To do this, we note that under the zero-discount rates condition, the expected present value of distress costs is given by the sum of the expected future distress costs cash flow, namely

$$E_0\left[V_t^{DC}\right] = \sum_{k=t+1}^{m} E_0\left[F_k^{DC}\right] \quad (16)$$

and that Equation (15) can be rewritten as:

$$f(t) = \sum_{k=t+1}^{m} \frac{E_0\left[F_k^{DC}\right]}{E_0[V_t]} \quad (17)$$

To keep distress costs rates much lower than unlevered rates and allowing for negative rates, the above condition can be modified as follows:

$$f(t) = \sum_{k=t+1}^{m} \frac{E_0\left[F_k^{DC}\right]}{E_0[V_{k-1}]} \quad (18)$$

in which the expected cash flow of distress costs at time k is weighted by the firm expected value at time k-1 and not at time t. Inserting Equation (18) into Equation (14), we get that distress costs discount rates can be expressed as described by the following assumption:

Assumption 2 :

$$k_t^{DC} = \ k_t^{U}(1 - \sum_{k=t+1}^{m} \frac{E_0\left[F_k^{DC}\right]}{E_0\left[F_t^{DC}\right]} \frac{E_0[V_t]}{E_0[k-1]}) \quad (19)$$

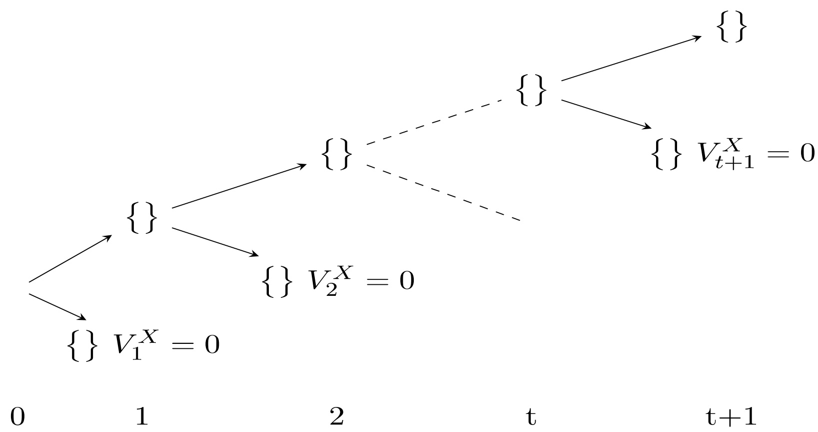
Assumption 2 can be interpreted as a kind of a *no-growth* condition, in the sense that it constrains distress costs discount rates to assume values well below the unlevered rates values. In fact, when distress costs rates reduce, the expected present value of distress costs cash flow, $E_0\left[V_t^{DC}\right]$, increases and, since $E_0\left[V_t^{DC}\right]$, appears in the denominator of the second term in Equation (7), it contributes to increase distress cost rates. The same reverting behavior happens when distress costs rates increase. The numerical analysis developed in the next Section confirms this behavior.

Substituting Equation (19) and Equation (12) into Equation (7), i.e., under Assumption 1 and Assumption 2, WACC rates become:

$$k_t^{W} = \left(1 + \sum_{k=t+1}^{m} \frac{E_0\left[F_k^{DC}\right]}{E_0[V_{k-1}]}\right) k^{U}{}_t - \frac{E_0[F_{t+1}^{TS}]}{E_0[V_t]} + \frac{E_0[F_{t+1}^{DC}]}{E_0[V_t]} \quad (20)$$

To model the stochastic process describing the time evolution of the firm value when default is allowed, we adopt a reduced form approach in which the default is treated as an *exogenous* event that can occur at any time during the lifetime of the firm. The valuation tree is shown in the following Figure 1.

Figure 1
The valuation tree



Curly brackets means a set of possible states of the world and the default event is represented by a downward sloped arrow. When default occurs, the debt holders obtain the firm value and the equity holders are left without anything. After the default, therefore, the equity $(V_t^X, X = S)$ as well as the debt value $(V_t^X, X = D)$ are equal to zero. This does not means that the recovery rate of the debt is zero. At default, on the contrary, the debt holders receive the firm value, i.e. the present value of the unlevered cash flow minus the present value of distress costs, and this value represents the recovery condition of the debt. This is the reason why we can set the debt value equal to zero in each state in which the firm defaulted.

Assumption 3. Let $p(t, I; t+1)$ the one-period survivorship probability, i.e. the probability that the firm is solvent at time t in the state i and remains solvent at time t+1, we assume that it is independent on the state i:

$$p(t, I; t+1) = p(t, t+1) = \frac{p(t+1)}{p(t)} \tag{21}$$

where p(t) is the unconditional survivorship probability at time t.

The last two assumptions of the model regard the structure of tax shields and the structure of distress costs. To determine the tax shields contribution, we assume that interests on debt are paid only in the states of the world in which the firm survives and that they are calculated on the outstanding debt at the nominal rate of the debt (Leland, 1994; Koziol, 2014). This means that the tax shields cash flow at time t+1 is related to the debt value at time t as described by the following assumption:

Assumption 4:

$$F_{t+1}^{TS} = T_c k^N V_t^D \qquad t = 0,1,2 \ldots, \tag{22}$$

in the states of the world at time t+1 in which the firm survives; $F_{t+1}^{TS} = 0$ in the states at time t+1 in which firm defaulted. T_c is the corporate tax rate and k^N is the nominal debt rate.

To model distress costs, we assume that they are proportional to the predistress firm value (Koziol, 2014). This assumption is motivated by the fact that data available on distress costs are expressed as a given percentage of the predistress firm value. Andrade and Kaplan (1998) estimate losses on the order of 10% to 23% of the predistress firm value and, in a more recent analysis, Glover (2016) confirms that the average costs among defaulted firms is about 25% of the predistress firm value. More specifically, we assume that distress costs at time t+1 are proportional to the firm value at time t as described by the following assumption:

Assumption 5: $$F_{t+1}^{DC} = aV_t \qquad t = 0,1,2 \ldots, \tag{23}$$

in the states of the world at time t+1 in which the firm defaulted; $F_{t+1}^{TS} = 0$ in the states at time t+1 in which the firm survives.

Under Assumptions 0-5, Equation (20) can be cast in the following form:

$$k_t^W = [1 + \alpha \sum_{k=t+1}^{m} (1 - \frac{p(k)}{p(k-1)})]\, k^U{}_t - T_c k^N \frac{p(t+1)}{p(t)} \frac{D}{V} + \alpha(1 - \frac{p(t+1)}{p(t)}) \tag{24}$$

Appendix B provides a detailed proof of Equation (24).

For practical purposes it may be convenient to use the following formula:

$$k_t^W = [1 + \alpha \cdot \ln\left(\frac{p(t)}{p(m)}\right)] k^U{}_t - T_c k^N \frac{p(t+1)}{p(t)} \frac{D}{V} + \alpha(1 - \frac{p(t+1)}{p(t)}) \tag{25}$$

in which the sum has been approximated with its continuous limit. Appendix B provides a detailed proof of Equation (25).

WACC formulas (24) and (25) depend on debt ratio D/V, but do not depend on the cost of the corporate debt. They account for tax shields and bankruptcy costs through the unconditional survivorship probability and in this way they include credit spread of corporate debt. Finally, we notice that Equations (24) and (25) are still valid also in the limit $m \to \infty$ if $\lim_{m\to\infty} p(m) \neq 0$. They are the main tool of analysis we will use in the next Section to investigate the firm valuation problem and the optimal capital structure.

IV. FIRM VALUE AND OPTIMAL CAPITAL STRUCTURE: A SIMULATION ANALYSIS

In this Section we propose a simulation analysis to illustrate the mathematical tractability of the model and its flexibility. The firm valuation is performed by discounting the unlevered cash flow at WACC rates (as given by Equations (24) and (25)) for different values of debt ratio and distress costs parameters. The optimal financial structure of the firm arises in a quite natural way as a consequence of the trade-off between the tax shields contribution to the firm value and the distress costs contribution. It is determined by the level of the debt ratio that maximizes the firm value. Furthermore, a numerical test on the validity of the main hypotheses of the model is provided.

In the following simulation we will assume that unlevered rates are constant at 10%, i.e. $k^U{}_t \equiv k^U = 10\%$, that the corporate tax rate is $T_c = 35\%$ and that the nominal rate of the debt is $k^N = 6\%$. We recall that $p(t)$ describes the probability of survivorship at time t and, of course, $1 - p(t)$ is the probability that default has occurred in the time interval $[0, t]$. For illustrative purposes, we assume that p(t) is modeled according to the following functional form:

$$p(t) = 1 - a(1 - e^{-bt}) \quad (26)$$

where

$$a = c \ \max[\frac{D}{V} - \left(\frac{D}{V}\right)_{th}, 0] \quad (27)$$

b and c are constant parameters and $(D/V)_{th}$ is a threshold value such that, for debt ratios D/V lower than $(D/V)_{th}$, the debt is risk-free (in such a case a=0 and $p(t) = 1$. In our simulation c is assumed equal to 1 an b equal to 0.1. This functional form of the survivorship probability has two main features: first, it depends on the debt ratio to account for the fact that higher debt ratios induce greater values of the default probability; second, the probability of survivorship at time t, in the limit t→ $+\infty$ is strictly positive, i.e., $\lim_{m\to\infty} p(t) = 1 - a$. Moreover, the functional form of the unconditional survivorship probability $p(t)$, when D/V is greater than $(D/V)_{th}$, depends on two parameters, namely b and c. Such parameters can be estimated for example by using the credit spread value of short term debt and long term debt. In this way the model can account for the whole firm's debt structure. Table 1 recaps the numerical values of the parameters used in the simulation analysis.

Table 1
Parameters assumptions

k^U	T_c	k^N	b	c
10%	35%	6%	0.1	1

Figure 2 shows the behavior of the firm value as a function of the debt ratio[3]. In the left panel, different curves refer to different distress costs hypotheses, namely $\alpha = 0.10, 0.15, 0.20, 0.30$ (the debt ratio threshold has been fixed at 20%). Such values have been chosen in agreement with empirical data reported in literature (Andrade and Kaplan, 1998; Glover, 2016). The optimal capital structure is then obtained when the debt ratio is about 70% in the case $\alpha = 0.10$. For increasing values of the distress costs, the optimum debt ratio reduces: to about 50% in the case $\alpha = 0.15$, 30% for $\alpha = 0.20$ and 20% (i.e. at the level of the riskless debt threshold) in the case $\alpha = 0.30$. In the right panel, the firm value is depx icted as a function of the debt ratio under three different hypotheses on the threshold riskless debt, namely $\left(\frac{D}{V}\right)_{th} = 0.10, 0.15, 0.20, 0.30$ (distress costs are kept at a constant level $\alpha = 0.15$). The optimal capital structure is obtained when the debt ratio is about 60% in the case $\left(\frac{D}{V}\right)_{th} = 0.30$. For decreasing values of the threshold debt ratio, the optimum reduces: to about 50% for $\left(\frac{D}{V}\right)_{th} = 0.20$ and about 40% in the case $\alpha = 0.10$. A similar behavior can be observed also in the case of a growing firm. Figure 3 shows the results of the numerical analysis in the case of a growing firm at constant rate $g = 3\%$.

Figure 2

Left panel: Firm value vs debt ratio at different levels of distress costs. Right panel: Firm value vs debt ratio at different levels of threshold debt ratio $\left(\frac{D}{V}\right)_{th}$

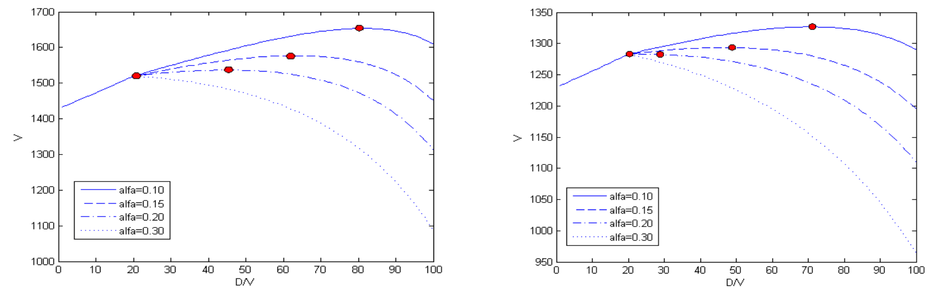


Figure 3

Growing firm at a constant rate g=3%. Firm value vs debt ratio at different levels of distress costs. Left panel: the infinite horizon case. Right panel: a 30 years horizon

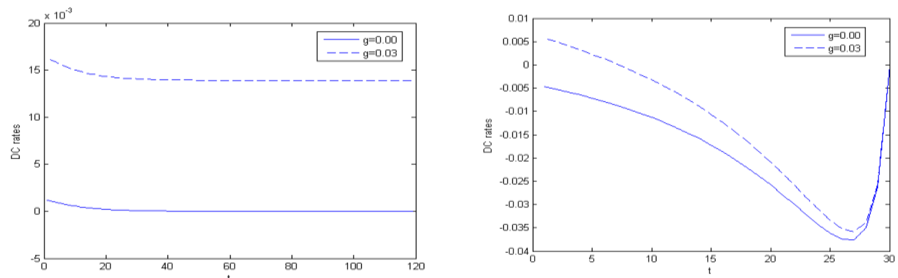
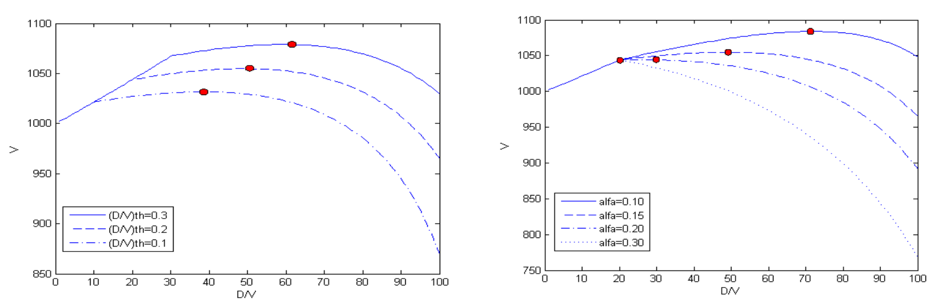


Figure 4

Distress costs rates when the debt ratio is $D/V = 0.15$, $\alpha = 0.15$ and the threshold level for riskless debt is $(D/V)_{th} = 0.2$. Left panel: The infinite time horizon case. Right panel: A 30 years time horizon



In the infinite time horizon case, discount rates show a decreasing behavior until they converge to a finite value in the limit $m \to \infty$. Some further comments are due in a finite time horizon context. In such a case, Equation (19) requires that $k^{DC}_{m-1} = 0$ at the end of the firm life, as it is easy to verify by posing $t = m - 1$. In the simulation analysis, distress costs rates show a decreasing behavior and can become negative. Due to the trade off between the increasing number of the state of the world in which the default event has occurred and the reduced variability of the distress costs as the end of operating life is approached, discount rates reach a minimum and then they increase until the end of the time horizon $(t = m)$ in which the condition $k^{DC}_{m-1} = 0$ must be fulfilled.

Finally, the last part of the simulation analysis is devoted to quantify the contribution of distress costs rates in the valuation process, i.e. to measure the impact of the third term in the r.h.s. of Equation (7) with respect to the last term of the same equation, $E_0\left[F^{DC}_{t+1}\right]/E_0[V_t]$. To do this, we provide a comparison between firm values obtained by using the WACC rates formula described by Equation (24) or (25) with those obtained by using WACC rates given by the following relationship:

$$\bar{k}^{W}_{t} = k^{U}_{t} - T_c k^{N} \frac{p(t+1)}{p(t)} \frac{D}{V} + \alpha\left(1 - \frac{p(t+1)}{p(t)}\right) \tag{28}$$

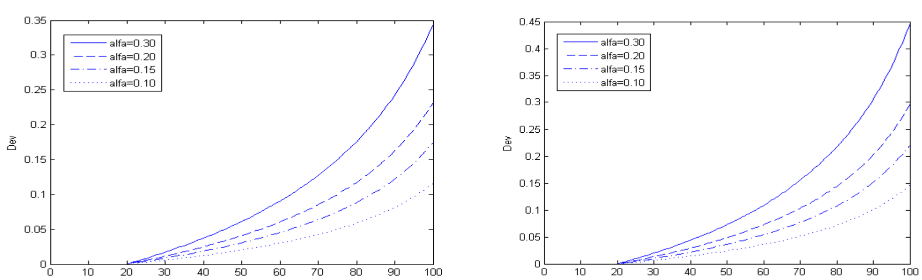
that can be derived from Equation (7) without considering the contribution of the third term and using the set of assumptions described in the previous Section. Let us define therefore,

$$\mathrm{Dev} \equiv \frac{|\bar{V} - V|}{V} \tag{29}$$

where $\bar{V}$ is the firm value calculated using WACC rates given by Equation (28). Figure 5 shows the behavior of Dev as a function of the debt ratio under four different hypotheses on the entity of distress costs, namely $\alpha = 0.10, 0.15, 0.20, 0.30$ (the debt ratio threshold has been fixed at 20%). The left panel depicts the behavior of Dev in the case of a no-growing firm; the right panel depicts the case of a growing firm at a constant rate $g = 0.3$.

Figure 5

Dev as a function of the debt ratio. Left panel: The case of a no-growing firm. Right panel: The case of a growing firm at a constant rate $g = 0.3$



Firm value variations are very sensitive to the debt ratio and to the entity of distress costs. As shown in Figure 5, such variations may be very relevant for debt ratio values greater than 50%. In the case of low debt ratio, if distress costs are not too high, Equation (28) offers a very interesting approximation that can be used in practical applications. In such cases, the main contribution to the present value of distress costs comes from the term $E_0\left[F_{t+1}^{DC}\right]/E_0[V_t]$. A similar formula has been proposed by Koziol (2014) under the hypothesis that the one period default probabilities, $1-p(t+1)/p(t)$, are constant over time. In such a case, WACC rates are independent on time and the valuation formula greatly simplify.

V. CONCLUDING REMARKS

The developed analysis aimed at putting in some evidence the possibility to build simple, flexible models to value companies under defaultable debt and bankruptcy costs within the context of DCF valuation techniques. One of the main result of the paper is to provide an analytic characterization of WACC rates in terms of the debt ratio, assumed constant over time, and of the one period survivorship probabilities. Although WACC rates result not constant over time, the firm valuation is straightforward and, from this point of view, the model seems to offer a good compromise between mathematical tractability and the capability of capturing many of the relevant financial aspects of the problem. Moreover, the optimal capital structure problem can be analyzed in term of the optimal debt ratio, i.e. the value of the debt ratio which maximizes the firm value. Within this context, the optimal capital structure depends on the firm activities cost of capital and, as a consequence, it is very sensitive to the industry sector. This fact suggests the possibility to test the model by analyzing the relation between the (average) debt ratio and the market cost of capital of different economic sectors. We leave this topic to future investigations.

ENDNOTES

1. The empirical literature on capital structure documented the existence of a target leverage and investigated the way in which firms adjust the leverage toward their target debt ratios (Byoun, 2008; Dang and Garret, 2015).
2. This is not surprising. For example, put options which can be used as hedging instruments when market returns are low, are characterized by negative betas (Cox and Rubinstein, 1985) and, depending on the magnitude of the beta value, by negative discount rates.
3. For illustrative purposes the nominal rate, k^N, has been assumed to be independent on the debt ratio. The model can be used also in the case in which the nominal rate is not constant but depends on the debt ratio.

APPENDIX A

This Appendix contains detailed proofs of the main results stated in Section II.

Proof of Equation (5). Let us explicitly rewrite Equation (3) in the five different specifications that follow:

$$\left(1+k_t^S\right)E_0\left[V_t^S\right]=E_0\left[F_{t+1}^S+V_{t+1}^S\right],$$

$$(1+k_t^D)E_0[V_t^D]=E_0[F_{t+1}^D+D_{t+1}],$$

$$(1+k_t^U)E_0[V_t^U]=E_0[F_{t+1}^U+V_{t+1}^U],$$

$$\left(1+k_t^{TS}\right)E_0\left[V_t^{TS}\right]=E_0\left[F_{t+1}^{TS}+V_{t+1}^{TS}\right],$$

$$\left(1+k_t^{DC}\right)E_0\left[V_t^{DC}\right]=E_0\left[F_{t+1}^{DC}+V_{t+1}^{DC}\right].$$

Let us sum the first and the second equation, then subtract the third and the fourth one, and finally, we sum the last equation, after some algebraic manipulations (in which Equations (2) and (1) have been used), we easily get:

$$k_t^S E_0\left[V_t^S\right]+k_t^D E_0[V_t^D]-k_t^U E_0[V_t^U]-k_t^{TS} E_0\left[V_t^{TS}\right]+k_t^{DC} E_0\left[V_t^{DC}\right]=0$$

Since $V_t^U=V_t-V_t^{TS}+V_t^{DC}$, the above equation can be rewritten in the following way:

$$k_t^E=k_t^U+(k_t^U-k_t^D)\frac{E_0[V_t^D]}{E_0[V_t^S]}-\left(k_t^U-k_t^{TS}\right)\frac{E_0[V_t^{TS}]}{E_0[V_t^S]}+(k_t^U-k_t^{DC})\frac{E_0[V_t^{DC}]}{E_0[V_t^S]} \qquad \text{(A.1)}$$

Proof of Equation (7). Let us consider the following set of recursive equations:

$$(1+k_t^W)E_0[V_t]=E_0[F_{t+1}^U+V_{t+1}],$$

$$(1+k_t^U)E_0[V_t^U]=E_0[F_{t+1}^U+V_{t+1}^U],$$

$$\left(1+k_t^{TS}\right)E_0\left[V_t^{TS}\right]=E_0\left[F_{t+1}^{TS}+V_{t+1}^{TS}\right],$$

$$\left(1+k_t^{DC}\right)E_0\left[V_t^{DC}\right]=E_0\left[F_{t+1}^{DC}+V_{t+1}^{DC}\right],$$

Subtracting from the first equation the second and the third one, and then summing the fourth, after some algebraic manipulations (in which Equations (2) has been used), we easily get:

$$k_t^W E_0[V_t]-k_t^U E_0[V_t^U]-k_t^{TS} E_0\left[V_t^{TS}\right]+E_0\left[F_t^{TS}\right]=0$$

Since $V_t^U=V_t-V_t^{TS}+V_t^{DC}$, the above equation can be rewritten as follows:

$$k_t^W=k_t^U-\left(k_t^U-k_t^{TS}\right)\frac{E_0[V_t^{TS}]}{E_0[V_t]}+\left(k_t^U-k_t^{DC}\right)\frac{E_0[V_t^{DC}]}{E_0[V_t]}-\frac{E_0[F_{t+1}^{TS}]}{E_0[V_t]}+\frac{E_0[F_{t+1}^{DC}]}{E_0[V_t]} \qquad \text{(A.2)}$$

Proof of Equation (8). Let us consider the following set of recursive equations:

$$(1+k_t^W)E_0[V_t]=E_0\left[F_t^{TS}+V_{t+1}\right],$$

$$(1 + k_t^S)E_0[V_t^S] = E_0[F^S{}_{t+1} + V^S{}_{t+1}],$$

$$(1 + k_t^D)E_0[V_t^D] = E_0[F^D{}_{t+1} + V^D{}_{t+1}],$$

Subtracting from the first equation the second and the third one, after some algebraic manipulations (in which Equations (2) and (1) have been used), we easily get:

$$k_t^W E_0[V_t] - k_t^S E_0[V_t^S] - k_t^D E_0[V_t^D] + E_0[F_{t+1}^{TS}] - E_0[F_t^{DC}] = 0$$

The above equation can be rewritten, therefore, as follows:

$$k_t^W = \frac{E_0[V_t^{TS}]}{E_0[V_t]} k_t^S + \frac{E_0[V_t^D]}{E_0[V_t]} k_t^D - \frac{E_0[F_{t+1}^{TS}]}{E_0[V_t]} + \frac{E_0[F_{t+1}^{DC}]}{E_0[V_t]} \qquad \text{(A.3)}$$

APPENDIX B

This Appendix contains detailed proofs of the main results stated in Section III.

Proof of Equation (24). To prove Equation (24), we must compute some averages, namely $E_0[F_{t+1}^{TS}]$, $E_0[V_t]$ and $E_0[F_{t+1}^{DC}]$. Under Assumption 4, we can express $E_0[F_{t+1}^{TS}]$ in the following way:

$$E_0[F_{t+1}^{TS}] = T_c k^N \sum_i \sum_{j_i} p(t,i)p(t,i;t+1,j_i)D_{t,i} \qquad \text{(B.1)}$$

where the sum over index i is restricted to the states at time t in which the firm is solvent and the sum over index j_i is limited to the states of solvency at time t+1 that can be reached by the state i at time t. $p(t,i)$ denotes the unconditional survivorship probability at time t in the state i, and $p(t,i;t+1,j_i)$ the one-period survivorship probability between the state i at time t and the state j_i at time t+1. Equation (B.1) can be rewritten as:

$$E_0[F_{t+1}^{TS}] = T_c k^N \sum_i p(t,i)D_{t,i} \sum_{j_i} p(t,i;t+1,j_i) = T_c k^N p(t,i;t+1) \sum_i p(t,i)D_{t,i} \qquad \text{(B.2)}$$

where the last equality can be obtained by Assumption 3:

$$\sum_{j_i} p(t,i,j_i) = p(t,i;t+1) = p(t;t+1) \qquad \text{(B.3)}$$

Since in each state i the debt ratio is constant, Equation (B.2) can be cast in the following final form:

$$E_0[F_{t+1}^{TS}] = T_c k^N p(t;t+1)\frac{D}{V}\sum_i p(t,i)\,V_{t,i} = T_c k^N p(t;t+1)\frac{D}{V}E_0[V_t] \qquad \text{(B.4)}$$

or, equivalently:

$$\frac{E_0[F_{t+1}^{TS}]}{E_0[V_t]} = T_c k^N \frac{p(t+1)}{p(t)} \frac{D}{V} \tag{B.5}$$

where p(t) is the unconditional survivorship probability at time t. In a similar way, we can obtain:

$$E_0[F_{t+1}^{DC}] = \alpha[1 - p(t; t+1)] \sum_i p(t,i) V_{t,i} = \alpha[1 - p(t; t+1)] E_0[V_t] \tag{B.6}$$

where Assumption 5 has been used. Equation (B.6) can be rewritten, therefore, in the following final form:

$$\frac{E_0[F_{t+1}^{DC}]}{E_0[V_t]} = \alpha \frac{p(t+1)}{p(t)} \tag{B.7}$$

Equation (24) can be obtained by substituting (B.5) and (B.7) into Equation (20):

$$k_t^W = [1 + \alpha \sum_{k=t+1}^{m} (1 - \frac{p(k)}{p(k-1)})] k_t^U - T_c k^N \frac{p(t+1)}{p(t)} \frac{D}{V} + \alpha \frac{p(t+1)}{p(t)} \tag{B.8}$$

Proof of Equation (25). To prove Equation (25), it suffices to approximate the sum in Equation (B.8) with its continuous limit:

$$\sum_{k=t+1}^{m} (1 - \frac{p(k)}{p(k-1)}) = - \sum_{k=t+1}^{m} (\frac{p(k)=p(k-1)}{p(k-1)} \cong - \int_t^m \frac{p\prime(x)}{p(x)} dx = \ln \frac{p(t)}{p(m)}$$

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