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Firm's Value under Investment Irreversibility, Stochastic Demand and General Production Function

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ABSTRACT

As mentioned in Pindyck (1988), since most investment expenditures are irreversible, investors can decide to allocate their amounts until the full value of an incremental unit of capacity equals its full cost. The investment process includes the option of using or not the bought units. We examine in particular the influence of both production functions and stochastic demand on capacity choice, firm's value and long-run marginal cost.

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I. INTRODUCTION

The theory of *real options* can potentially provide fundamental analysis of investment values: research and development projects, natural-resource investments, technological innovations...Besides, real options have several characteristics in common with financial options[1]: in particular, they can take account of other investment opportunities such as standard financial assets. Using a substitute asset, a risk-premium can be determined.

Pindyck (1988) introduces a model to value firm or investment project, based on an incremental investment process which allows to buy capital units tick-by-tick. This kind of process is particularly interesting when investments are supposed irreversible: the amount of invested capital can be controlled according to the actual demand or market price for each unit of output. More generally, the investor can benefit at any time from available information. However, he cannot disinvest whatever the future evolution of the market. This implies that the value of the unit invested must be significantly higher than the sum of the purchase and of the production cost. In fact, the option to invest or not can be valued according to other investment opportunities[2]. Besides, each unit of capacity allows potentially the firm to choose to produce or not at each future time. Each of these opportunities is an option with strike equal to the production cost.

In this framework, an optimal capacity can be determined according to market price and production cost. The optimality corresponds to the equality of the present value of the expected cash flow from a marginal unit of capacity with the total cost of this unitTo determine this optimal value, first we have to evaluate each additional unit of capacity and, second, to price the option to invest in this unit. It is assumed that if demand is too low, the firm can decide not to use this invested amount.

In this paper, we propose to extend this analysis into two main directions: First, we introduce a more general process to describe the market value of the output. This allows in particular to take account of mean reverting properties which have been previously emphasized from some empirical data (see Quigg (1993, 2001) for example). We can suppose also that the market price of each unit of output can be non linearly modified by the level of the total output. Second, we consider potentially more general productions functions to describe the firm's activity. For example, for each capital input, the corresponding output may be a non linear function of this input.

This paper is organized as follows: Section II presents the general model of firm's production. We introduce precise assumptions in particular on the cost production function and a market price variation with respect to the level of output that is proposed by the firm. Section III presents the investment and firm's valuation. We show how it depends on the production function and on the dynamics of the market price (all technical proofs are available on request). Section IV illustrates numerically the previous theoretical results.

II. THE PRODUCTION OF THE FIRM

According to the firm's technological possibilities, the production set is determined from production activities. As usual, the goal of the firm is the profit maximization which indicates a corresponding supply. We assume that there exists basically one kind of output[3]. This is the most frequently encountered production model. It is generally described by means of a production function that gives the maximum amount of the output that can be produced using the input. Besides, we suppose also as usual that the production set is convex which is equivalent here to the concavity of the production function.

Consider a firm that faces the following cost and production constraints:

- Each unit of capital is bought at a cost K_t per unit.
- For each capital stock K_t, the corresponding output Q_t that can be produced per time period is bounded by $Q^M(t, K_t)$ which is a concave function of the input K_t.
- The firm has an operating cost denoted by $C_t(Q_t)$. This cost is assumed to be an increasing function of the production level[4]. Since the production function is supposed to be concave, the cost function C is convex.
- We suppose that the price of the output per unit is given by:

$$P(t, \theta_t, Q_t) = \theta_t - P(t, Q_t)$$

where θ is the price per unit when the firm is price-taker. It evolves [5] as follows:

$$d\theta_t / \theta_t = \alpha(t, \theta_t)dt + \sigma(t, \theta_t)dz_{\theta,t} \quad (1)$$

where the instantaneous mean $\alpha(\theta)$ and variance $\sigma^2(\theta)$ are supposed to satisfy usual condition to guarantee existence and uniqueness of the solution of Equation (1)[6]. The term dz_θ is an increment of a standard Wiener process. Note that the mean $\alpha(.)$ and the variance $\sigma^2(.)$ can be estimated from statistical data.

The process θ represents the demand shift parameter and as usual, we have $\frac{\partial P(t, \theta, Q)}{\partial \theta} > 0$ $(=1,$ here). We assume that the price variation P, due to the production level Q, is increasing with respect to Q. The price P is an increasing function of θ and a decreasing function of the output[7].

Definition: Denote by $Oc(t, Q)$ the marginal operational cost function: Q

$$Q \to Oc(t, Q) = P(t, Q) + Q\frac{\partial P}{\partial Q} + \frac{\partial C}{\partial Q} \quad (2)$$

Assumption: We suppose that the function $Oc(t, Q)$ is increasing[8].

Definition: Let $\theta^M(t, K_t)$ be the value of the demand θ such that the optimal output for this demand level is the maximal output $Q^M(t, K_t)$ for the capacity K_t.

Proposition (Optimal output for a given capacity)

$$\theta^M(t,K_t) = P(t,Q^M(t,M_t)) + Q^M(t,K_t)\frac{\partial P}{\partial Q}(t,Q^M(t,K_t) + \frac{\partial C}{\partial Q}(t,Q^M(t,K_t))$$

Proof: The investor solves the following optimization problem:

$$\text{Max}_Q\left[QP(t,\theta_t,Q_t) - C(t,Q_t)\right]$$

The first order condition gives:

$$\theta_t - \left[P(t,Q) + Q\frac{\partial P}{\partial Q}(t,Q) + \frac{\partial C}{\partial Q}(t,Q)\right] = 0 \quad (3)$$

The value $\theta^M(t,K_t)$ is the solution of previous Equation (3) when the output Q is maximal at the capacity level K

An example: consider the following case:

- $Q^M(t,K_t) = K_t$
- $P(t,Q) = \varphi Q^p$, where φ and p are positive constants and p>0.
- $C(t,Q_t) = \psi Q^m$, where ψ and m are positive constants and m>1.

In that case, the function $Oc(t,Q)$ is equal to $Oc(t,Q) = \varphi(1+p)Q^p + \psi m Q^{m-1}$. Thus, $Oc(t,Q)$ is increasing.

III. THE FIRM'S VALUE

A. Spanning by Existing Assets

We assume that stochastic changes in demand are spanned by existing assets, so there is an asset or a portfolio of assets whose price is perfectly correlated with θ . It means that the investors' opportunities are not affected by the firm's decision to invest or produce. We can apply this hypothesis to the most commodities traded on both spot and futures markets and to manufactured goods whose prices are correlated with the values of portfolios. The spanning assumption will allow us to determine the investment rule that maximizes the firm's market value.

Let x be the price of an asset or portfolio of assets perfectly correlated with θ , and $\rho_{\theta m}$ be the correlation of θ with the market portfolio. Then x will be expressed by the following equation:

$$dx = \mu x dt + \sigma x dz,$$

where, according to the CAPM, μ is the expected return expressed by:

$$\mu = r + \Phi\rho_{\theta m}\sigma$$

and Φ is the market price of risk. In what follows, we suppose that the expected percentage rate α of change of θ, is smaller than μ Denote the difference between μ and α by δ, that is: $\delta = \mu - \alpha$

B. The Investment Process

1. Optimal Incremental Investment Decisions

Assume, as in Pindyck (1988), that the firm can invest units of capital one at a time, at a sunk cost k per unit, whenever it wishes. Let k be the amount of capital currently in place, the firm's value W will be expressed as the sum of two parts:

$$W = V(K;\theta) + F(K;\theta), \tag{4}$$

where $V(K;\theta)$ is the value of the firm's capital which is the present value of the expected flow of profits that will be generated by the capital, given the current value of θ and $F(K;\theta)$ is the present value of any additional profit that might result if the firm decide to add more capital in the future. $F(K;\theta)$ represents the value of the firm's "growth option".

Due to the fact that the firm is not committed to any investment path, we can observe that $F(K;\theta)$ is greater than the present value of the expected flow of net profits, generated by the anticipated future investments.

From Equation (4), we can examine the firm's investment problem by noting that *units of capital are invested sequentially.* Suppose now that the units are discrete: we can number them in the order they are invested. Suppose also that units 1 to n have been invested so far. The following equation results from the previous one Equation (4), by deleting θ and by summing the value of each invested unit and the values of the options to invest further units.

$$W = \Delta V(0) + \Delta V(1) + ... + \Delta V(n-1) + \Delta F(n) + \Delta F(n+1) + ... \tag{5}$$

Note that $\Delta V(j)$ is the value of the j+1 unit of capital. It is the present value of the expected flow resulting from the incremental profits generated by unit j+1.

- The firm can use this unit of capital at every moment.
- The firm has to decide to invest or not an additional capital. The amount $\Delta F(n)$ is the value of the option to buy one more unit n+1 at any time in the future. If the firm decides to exercise this option, it pays k and will receive an asset worth $\Delta V(n)$
- The firm also gives up $\Delta F(n)$ because once exercised, the option is out.
- It will not be possible to disinvest (whether or not the firm later buys more capital) and now it has paid for unit n+1.
- $\Delta F(n)$ is the cost of investing in this unit, $k + \Delta F(n)$ is the full cost of investing

and it has to be compared to the benefit $\Delta V(n)$

- After buying unit n+1, the firm has to decide at what point it has to exercise its option, worth $\Delta F(n+1)$, to buy unit n+2, which is worth $\Delta V(n+1)$ and so on.

These options must be exercised sequentially. The option to grow will be expressed as followed:

$$F(n) = \sum_{j=n}^{\infty} \Delta F(j)$$

If we let these incremental units become infinitesimally small, Equation (5) can be written as follows:

$$W = \int_0^K \Delta V(v;\theta)dv + \int_K^{\infty} \Delta F(v;\theta)dv, \tag{6}$$

where $\Delta V = \frac{\partial V}{\partial K}$ and $\Delta F = \frac{\partial F}{\partial K}$

The firm's optimal capital stock K^* maximizes the net value W-kK^*. Using Equation (6), this maximization gives the following first order optimality condition that must hold if the firm is investing:

$$\Delta V(K^*;\theta) = k + \Delta F(K^*;\theta) \tag{7}$$

2. Value of a Marginal Unit of Capacity

Definition (Stochastic profit) The stochastic profit per unit $\Delta\pi_t$ is given by:

$$\Delta\pi_t(K_t) = \text{Max}\left[0, (\theta_t - \theta^M(t, K_t))\right]$$

Thus, the marginal value per unit $\Delta V(K)$ satisfies:

$$\Delta V(K) = \int_0^{\infty}\int_0^{\infty} \Delta\pi_t(\theta, K) f(\theta, t) d\theta e^{-\mu t} dt, \tag{8}$$

where $f(\theta, t)$ is the density at time t and μ is the risk-neutral discounted rate.

Brennan and Schwartz (1985), Mc Donald and Siegel (1985) had studied the valuation of a factory that can be temporarily shut down. Note that from Equation (8), the present value of an incremental profit at future time t is the value of a European call option, with expiration date t and exercise price $\theta^M(t, K_t)$, on a stock whose price is θ, paying a proportional dividend δ. When we sum up the values of these call options for every future t, we obtain $\Delta V(K)$. However, as mentioned in Pindyck (1988)), this does not provide a closed form solution.

Thus, using a similar approach as in Brennan and Schwartz (1985), we search for the value of a firm that produces one unit of output per period with operating cost $Oc(t, M(t, K_t))$ which is sold at a given price θ_t Note that, as in Pindyck (1988), we assume that the firm can be shut down costlessly during a time period as soon as θ_t is smaller than the operating cost.

We determine the present value of the expected flow of profits from an incremental unit of capacity for a given capacity K
Recall the Pindyck's solution for the geometric Brownian motion case:

$$\text{If } \theta \le \theta^M(t, K_t): \qquad \Delta V_t(K_t) = b_1 \theta^{a_1}$$

$$\text{If } \theta \ge \theta^M(t, K_t): \qquad \Delta V_t(K_t) = b_2 \theta^{a_2} + \frac{\theta}{\delta} - \frac{\left\lfloor \theta^M(t, K_t) \right\rfloor}{r}, \tag{9}$$

where

$$a_1 = -\left(\frac{r - \delta - \frac{a^2}{2}}{\sigma^2}\right) + \frac{1}{\sigma^2}\left[(r - \delta - \frac{\sigma^2}{2})^2 + 2r\sigma^2\right]^{\frac{1}{2}} > 1,$$

$$a_2 = -\left(\frac{r - \delta - \frac{a^2}{2}}{\sigma^2}\right) + \frac{1}{\sigma^2}\left[(r - \delta - \frac{\sigma^2}{2})^2 + 2r\sigma^2\right]^{\frac{1}{2}} < 0,$$

and

$$b_1 = \frac{r - a_2(r - \delta)}{r\delta(a_1 - a_2)}\left[\theta^M(t, K_t)\right]^{1-a_1} > 0,$$

$$b_2 = \frac{r - a_1(r - \delta)}{r\delta(a_1 - a_2)}\left[\theta^M(t, K_t)\right]^{1-a_2} > 0,$$

The assumption that the process θ follows a geometric Brownian motion means that θ has a determinist trend but for a long horizon, θ may have values very far from this prediction. On the contrary, if past observations of prices of the output (or similar products) allow to induce a long-term value with a good prediction, then we must rather introduce mean-reverting-process to model the θ dynamics.

In what follows, we provide the solution when $(\theta)_t$ is a mean-reverting process:

$$d\theta_t = (\hat{a} - \hat{b}\,\theta_t)dt + c(t, \theta_t)dz_{\theta t},$$

where $\hat{a}$ and $\hat{b}$ are two positive constants and the function $c(t, \theta)$ is either a constant c (case of an Ornstein-Uhlenbeck process) or a square root: $c(\theta)=\sqrt{\theta}$ (case of a CIR type process).

Proposition (Value of the Expected Flow of Profits with Mean-Reverting Price)

(a) The Case of an Ornstein-Uhlenbeck Process (OU)

Introduce the function

$$i(\theta)=\sqrt{-\tilde{a}^2-2\tilde{a}\tilde{b}\theta-\tilde{b}\theta^2+4\tilde{c}}$$

where $\tilde{a}=-2\dfrac{a}{c^2}$, $\tilde{b}=2\dfrac{r-b}{c^2}$, $\tilde{c}=-2\dfrac{r}{c^2}$.

Define the functions $f_{1,0}(\theta)$ and $f_{2,0}(\theta)$:

$$f_{1,}(\theta)=\exp\left[\frac{1}{2}\left(\left(\tilde{a}+\tilde{b}\theta\right)(-\theta)+2\text{Log}\left[\text{Cos}\left[\frac{1}{2}i(\theta)(-\theta)\right]\right]\right)\right],$$

$$f_{2,0}(\theta)=2f_{1,0}(\theta)\left[\frac{\text{Sin}[i(\theta)\theta/2]}{i(\theta)}\right],$$

For the OU case, the general solution $\Delta V(\theta)$ is given by:

If $\theta_t\le\theta_t^M$: $\Delta V(\theta)=\alpha_1 f_{1,0}(\theta)+\alpha_2 f_{2,0}(\theta),$

where α_1 and α_2 are constant parameters.

If $\theta_t>\theta_t^M$: $\Delta V(\theta)=\beta_1 f_{1,0}(\theta)+\beta_2 f_{2,0}(\theta)+\left(\dfrac{a/b+\theta^M}{r}\right)+\left(\dfrac{1}{b}\right)\theta,$

where β_1 and β_1 are constant parameters.

(b) The Case of a Cox-Ingersoll-Ross Process (CIR)

Denote by Φ the Hypergeometric1F1 function. Define the functions $f_{1,c}(\theta)$ and $f_{2,0}(\theta)$:

$$f_{1,c}(\theta) = \Phi\left[-\frac{r}{r-b}, -2a, -2(r-b)\theta\right] + \Phi[1 + 2a - r/(r-b), -2(a-1)b, -2(r-b)\theta].$$

$$f_{2,c}(\theta) = f_{1,c}^2(\theta)\exp[2(a-(r-b)\theta)\text{In}[\theta]]$$

For the CIR case, we deduce that the general solution $\Delta V(\theta)$ is given by:

If $\theta_t \le \theta_t^M$: $\Delta V(\theta) = y_1 f_{1,c}(\theta) + y_2 f_{1,c}^2(\theta)\exp[2(a-(r-b)\theta)\text{In}[\theta]],$

where γ_1 and γ_2 are constant parameters.

If $\theta_t > \theta_t^M$:

$$\Delta V(\theta) = \delta_1 f_{1,c}(\theta) + \delta_2 f_{1,c}^2(\theta)\exp[2(a-(r-b)\theta)\text{In}[\theta]] + \left(\frac{a/b - \theta^M}{r}\right) + \left(\frac{1}{b}\right)\theta,$$

where δ_1 and δ_2 are constant parameters.

Examine the solution $\Delta V_t(K_t)$:

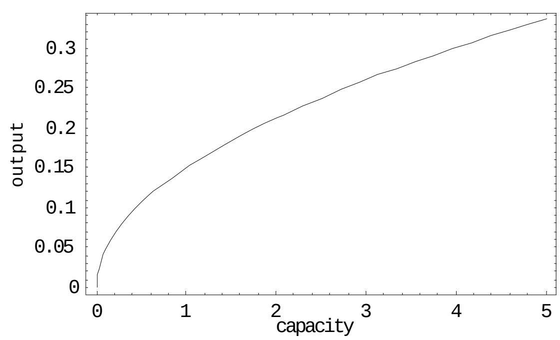
- When $\theta < \theta^M(t, K_t)$, the unit of capacity is not invested since it cannot be used to produce. Then $\Delta V_t(K_t)$ represents the value of the firm's option to utilize the unit in the future, should θ increase.
- When $\theta \ge \theta^M(t, K_t)$, the unit is invested. If irrespective of changes in θ, the firm had no choice but to continue utilizing the unit throughout the future, the present value of the expected flow of profits would be given by $\frac{\theta}{\delta} - \frac{\lfloor \theta^M(t, K_t) \rfloor}{r}$ However, if θ falls, the firm can reduce output and not make use of this unit of capacity.
- Keeping K constant, we note that $\Delta V(\theta)$ looks like the value of a call option. In fact it is the sum of an infinite number of European call options. Therefore, as a Call option, ΔV increases with the volatility σ (for the Geometric Brownian motion case, see Pindyck (1988)). For $\sigma > 0$, $\Delta V_\sigma(K) > \Delta_{\sigma=0}(K)$ because the firm has not to use its capacity. When $\theta \to \infty$, $\Delta V(K) \to \Delta V_{\sigma=0}(K)$,For θ very large relative to K , this unit of capacity will almost be continuously used over a long period of time.
- Examine now the evolution of $\Delta V(K)$ (keeping θ constant): due to the stochastic evolution of the demand, a marginal unit of capacity has some positive value independently of the size of the existing capital stock. The greater is σ, the more slowly $\Delta V(K)$ decreases with K. Also, the smaller is K, the more likely it is that the marginal unit will be utilized, and so the smaller is $\Delta V(K) - \Delta V_{\sigma=0}(K)$ When K=0, $\Delta V(0) = \Delta V_{\sigma=0}(K)$; with C_1=0, the first marginal unit will always be used.

To illustrate the properties, consider the following case:

- $Q^M(t, K_t) = vK_t^q$ where v and q are positive constants.
- $P(t, Q) = \varphi Q^p$ where φ and p are positive constants.
- $C(t, Q_t) = \psi Q^m$ where ψ and m are positive constants and m>1.

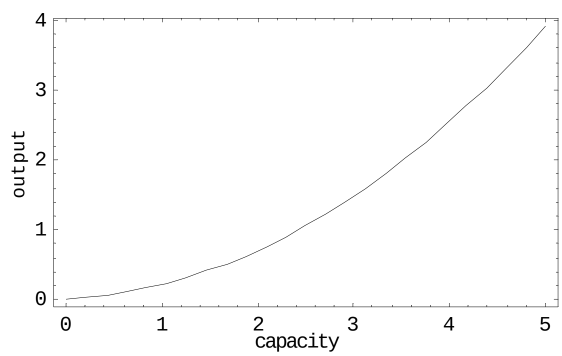
In that case, we have $\theta^M(t, K_t) = \varphi(1+p)v^p K_t^{qp} + \psi m v^{(m-1)} K_t^{q(m-1)}$. For $\varphi = 1/20$, $\psi = 1/20$, u = 1, p = 1/2, q = 1, m = 3, the function $\theta^M(t, K_t)$ has the following graphic, according to parameters values :

Case $m = 1.5$:



Value of θ for a complete use of the capacity (concave)

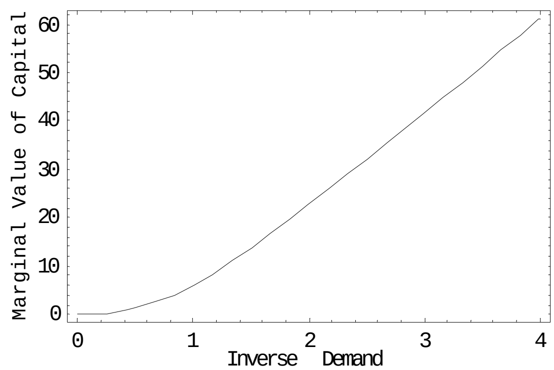
Case $m = 3$:



Value of θ for a complete use of the capacity (convex)

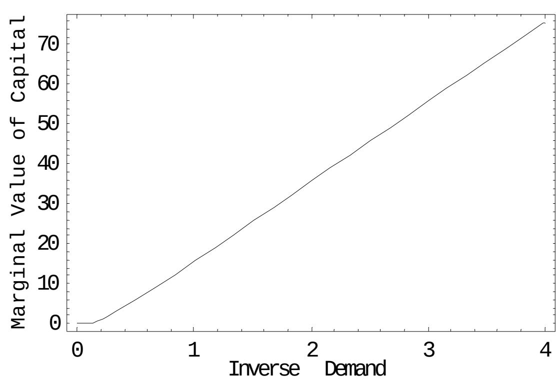
We examine now the optimal capacity $\Delta V(K^*)$ as function θ for different production functions (*Geometric Brownian motion case):* Parameter values: K=2.6 (then $\theta^M(K)=1$), y=0, c_1=0, c_2=1, $\delta = 0.05$, $\sigma = 0.2$, $r = 0.05$. Thus, we have: a_1=2.15831, a_2=-1.15831, b_1=6.03023, b_2=6.03023.

Case m=1.5



ΔV as a function θ of for K with $\theta^M(K)=1$ (convex)

Case m=3: (the more convex the cost function, the “more” linear the marginal value of capital)



ΔV as a function θ of for K with $\theta^M(K)=1$ (quasi linear)

Proposition (Value of the Expected Flow of Profits)

We determine the present value of the expected flow of profits from an incremental unit of capacity for a given capacity K. Recall the Pindyck’s solution for the geometric Brownian motion case:

If $\theta \le \theta^*(t, K_t)$: $\Delta F_t(K_t) = a\theta^{a1}$,

If $\theta \ge \theta^*(t, K_t)$: $\Delta F_t(K_t) = \dfrac{a_2 b_2}{a_1}\theta^{a2} + \dfrac{1}{\delta a_1}\theta$ (10)

Denote $a = \dfrac{a_1 b_2}{a_1}(\theta^*)^{(a_2 - a_1)} + \dfrac{1}{\delta a_1}(\theta^*)^{(1-a_1)} > 0$, a_1, a_2 and b_2 are given in Equation (10). The value $\theta^*(K)$ is the critical value of θ at or above which it is optimal to purchase the marginal unit of capacity, that is, the firm should purchase the unit if $\theta \ge \theta^*(K)$ Thus, the critical value of $\theta^*(K)$ is given by:

$$\frac{b_2(a_1 - a_2)}{a_1}\theta^*)^{a_2} + \frac{(a_1 - 1)}{\delta a_1}\theta^* - \frac{\theta^M(t, K_t)}{r} - k = 0 \tag{11}$$

(a) The Solution for the Ornstein Uhlenbeck Case is Given by:

If $\theta_t \le \theta^*_{OU}(K)$: the value of $\Delta F(\theta)$ is equal to

$$\Delta V(\theta) = \alpha_{1,F} f_{1,0}(\theta) + \alpha_{2,F} f_{2,0}(\theta),$$

where $\alpha_{1,F}$ and $\alpha_{2,F}$ are constant parameters.

If $\theta_t > \theta^*_{OU}(K)$: the value of $\Delta F(\theta)$ is given by

$$\Delta F(\theta) = \Delta V(\theta) - k$$

(b) The Solution for the Cox-Ingersoll-Ross Case is Given by:

If $\theta_t \le \theta^*_{CIR}(K)$: the value of $\Delta F(\theta)$ is given by

$$\Delta V(\theta) = y_{1,F} f_{1,c}(\theta) + y_{2,F} f^2_{1,c}(\theta) \exp[2(a - (r - b)\theta)\mathrm{In}[\theta]],$$

where $y_{1,F}$ and $y_{2,F}$ are constant parameters.

If $\theta_t > \theta^*_{CIR}(K)$: the value of $\Delta F(\theta)$ is given by $\Delta F(\theta) = \Delta V(\theta) - k$.

Note that all previous equations $\Delta F(\theta) = \Delta V(\theta) - k$ can be solved numerically for θ^*.

3. The Firm's Optimal Capacity

The capacity value $K^*(\theta)$ maximizes the firm's market value, net of cash outlays for the purchase of capital. Recall that the firm's net value as a function of its capacity K is given by $W(K)$.

The net value W of the firm as function of its capacity K is given by:

$$W(K) = \int_0^K \Delta V(v;\theta)dv + \int_K^\infty \Delta F(V;\theta)dv - kK.$$

Denote by K^* the optimal value. Differentiating with respect to K shows that this is maximized when K^* is such that:

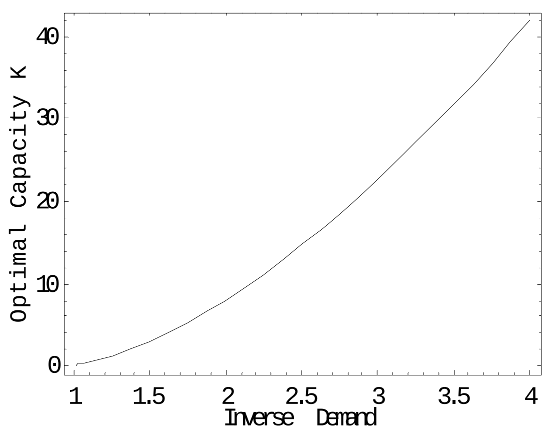
$$\Delta V(K^*) = \Delta F(K^*) + k. \tag{12}$$

We have to solve Equation (12). Except for special parameters values, this equation has no explicit solution. However, standard numerical methods allow to get the solution.

In all cases, the optimal capacity K^* is an increasing function of the inverse demand θ. In what follows, we illustrate how it depends on the cost production function.

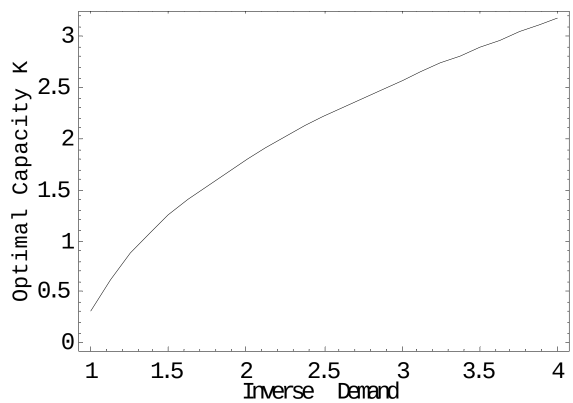
Geometric Brownian motion case:

Case m=1.5



Optimal capacity as function of θ (convex)

Case m=3. The optimal capacity is a concave function of θ as soon as the cost function is "sufficiently" convex.



Optimal capacity as function of θ (concave)

IV. SUMMARY

The options to develop real assets such as goods or services cannot be valued such as usual financial assets. This is based on the fact that the investment process may require to improve information about for example the possible aggregate demand, the characteristics of customers, the potential rivals. In this framework, Pindyck (1988) proposes a model based on an incremental investment process. We analyze a more general model taking account of other possible dynamics of real assets and production functions which allows us to analyze main of the industries. Such results can allow to better measure the performance and risk of firm's investment. Thus, a more accurate value of firms can be deduced.

ENDNOTES

1. We refer for example to Dixit and Pyndick (1994), Trigeorgis (1996) or Schwartz and Trigeorgis (2001) for general properties about real option analysis. Other references are for example Brennan and Schwartz (1985) for valuation of natural resources investments, Paddock and al. (1988) for the case of offshore petroleum leases, McDonald and Siegel (1986) for the value of time to invest.
2. As mentioned in Pyndick (1988), such property has been previously examined in McDonald and Siegel (1986): they prove that the value of such option can be large and must be taken into account in order to get a good estimation of real projects. Brennan and Schwartz (1985), Majd and Pindyck (1987), and others prove also that the levels of uncertainty are crucial to give a correct evaluation.
3. Of course, it is possible to introduce several kinds of outputs (see for example, Gale (1960), McFadden (1957), Champsaur and Milleron (1983)).

4. In Pindyck (1988), this cost function is quadratic: $C(Q) = c_1 + 1/2,\ c_2 Q^2$.
5. In Pindyck (1988), the factor θ is supposed to be a geometric Brownian motion. Our generalization allows for example to model mean-reverting effects.
6. See for example Jacod and Shiryaev (1997) for such standard conditions. Note that, under these assumptions, the process θ is a Markov process.
7. In Pindyck (1988), the price of output per unit is: $P(t, \theta_t, Q_t) = \theta_t - yQ_t$
8. For example, this hypothesis is satisfied as soon as both compensation price P and cost function C are convex.
9. As usual, we consider a portfolio which is long the value we have to calculate as function of the underlying "asset" and short "Delta" of this underlying "asset".
10. See Appendix of Pindyck (1988) when the process θ follows a Geometric Brownian motion.
11. We assume that the function μ has the same form as α, with $\mu > \alpha$ Thus, there exists two constant positive parameters a and b such that $\delta(\theta) = a/\theta + b$.

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