INTERNATIONAL JOURNAL OF BUSINESS, 3(2), 1998 ISSN: 1083-4346

Portfolio Optimization Under Realistic Short Sales Restrictions

Manuel Tarrazo and Gregory Alves

This study reviews and clarifies Alexander's pioneer contributions to the optimization of an all-equity portfolio under realistic short-sales restrictions. We also present a mixed-integer variable model that supports Alexander's economic rationale of the restricted short-sales methodology through a comprehensive and rigorous numerical-analysis framework. Our mixed-integer formulation captures the dual nature of the variables involved in realistic portfolio optimization and preserves the consideration of restrictions at the individual security level as required by practice. A single index-based formulation, which includes an example, is used throughout this paper.

I. INTRODUCTION

This communication was motivated by Alexander's seminal contributions [2, 3]. Our two main goals are to clarify how to obtain Alexander's [3] results and to introduce a mixed-integer optimization model that captures the dual nature of the variables involved in realistic portfolio optimization while preserving the consideration of restrictions at the individual security level as required by practice.

There is little doubt that portfolio theory is perhaps the most important contribution of Finance to Economics. Portfolio theory was developed by Harry Markowitz during the 1950's [15] and was the result of the author's multidisciplinary effort in economics, finance, mathematics, and operations research, fields which portfolio theory itself has enriched ever since. A group of distinguished researchers then investigated the aggregate implications of a market populated by Markowitz-type investors. The result was the development of theoretical equilibrium formulations for security markets, such as the Capital Asset Pricing Model --e.g., Sharpe's [18] -- which spanned practical models

Manuel Tarrazo, Finance Department, McLaren School of Business, University of San Francisco, San Francisco, CA 94123. **Gregory Alves**, Hutchinson Richardson Investment Management, LLC, 235 Montgomery St., Suite 725, San Francisco, CA 94104.

such as the single index model. Two whirlpools of activity dominated the 1970's: a) theoretical pricing of options, and b) the explosion of managed funds, whose rationale for existence is traceable to a single word related to portfolio theory: diversification. Hedging and portfolio insurance were some of the dominating themes of the 1980's. Alexander's [2, 3] research on the optimization of a portfolio under realistic short-sales restrictions may be regarded as one of the most significant contributions to portfolio theory during the 1990's, both because it provides a framework for understanding the role of hedging when both cash and short selling are possible and because it provides a reasonable solution for practical investment management.

Selling short (or short selling) refers to the sale of a security not owned by the seller. Short selling deserves consideration for important practical reasons. For example, short selling can enhance the return of a portfolio because security prices fluctuate, and it can also contribute to diminishing the risk of a portfolio of positively related securities, according to Renshaw [16]. Moreover, short sales are also an alternative to "indexing" or holding replicas of the market portfolio, according to Treynor and Black [20]. From a theoretical point of view, short sales embody hedging strategies that are necessary to obtain equilibrium in security markets via arbitrage relationships, as stated by Sharpe [18]. Consequently, it is not surprising that alternative definitions of short sales have received a great deal of attention from both academia and practitioners ever since the origins of modern portfolio theory.

The literature on portfolio optimization distinguishes two alternative versions of short-selling restrictions, the standard and Lintner's cases. In the standard case, no margin deposits are required; the only requirement in optimal weights is that they add up to "one". Furthermore, the investor is able to use the proceeds from short selling to finance the purchase of other securities. In Lintner's case, there is a margin of 100% on short positions, which means the investor must purchase a "bond" of the same amount as the short sale to pay the riskless rate of interest to the investor. However, many of the formalized efforts have ignored the realistic case in which the short seller faces a margin requirement of less than 100% on his shorted positions and in which the short seller may obtain interest on both margin deposits and proceeds but at different rates. Moreover, these requirements are applied at the security level, not at the portfolio level.

Alexander [2] provides a methodology for dealing with the realistic short-sales case. His methodology contains both the economic rationale and the corresponding optimization technique for obtaining optimal portfolio weights. Alexander [3] provides an example of his optimization technique, which is based on the Elton-Gruber-Padberg [6, 7, 8, 9] algorithm. This study supports Alexander's [3] economic rationale of the restricted short-sales methodology through a more comprehensive and rigorous numerical analysis framework.

The first section of this paper provides clarification on how to obtain Alexander's results. A mixed-integer, binary variable model is introduced in the second section. It is shown that quadratic mixed-integer optimization of our model reproduces Alexander's results. Concluding comments close the paper.

II. ALEXANDER'S METHODOLOGY

As we noted earlier, selling short refers to the sale of securities not owned by the seller. In the standard short-sales case, these sales represent a source of funds that alleviate the budget constraint of the investor and are represented as negative weights in portfolio optimizations. Moreover, these negative weights may also contribute to diminishing portfolio risk under certain conditions, according to Renshaw [16].

In the standard case presented in textbooks, e.g., Sears and Trennepohl [17, Chapter 7], optimal security weights can take on negative values, and these negative weights are not restricted in size. Markowitz [13,14,15] has always been suspicious of this treatment of short sales; he excluded this possibility in his contributions to portfolio theory, which reflect his priority of providing rigorous but practical investment guidelines to individual investors, many of whom may elect alternative hedging techniques to short selling (e.g. cash, fixed income). As he notes, the standard short-sales case in which portfolio weights are only required to add up to one "... allows the investor to place bets of arbitrarily large size with potential losses that he cannot cover himself" [13, p. 289]. Sharpe [18] discusses the effects of trading restrictions on the CAPM model.

Lintner's [12] analysis, in which short sellers would be required to deposit a 100% margin of their shorted positions and would receive interest payments on both the margin deposits and on the short sales proceeds at the same risk-free rate, incorporated some realism into the analysis but still cannot not be used in practical endeavors.

In practice, investors especially institutional investors do have access to short sales. Such investors may or may not have access to interest payments on the margined positions (rf_1) or on the short-sales proceeds (rf_2) -- which generally pay a risk-free rate equal to or lower than rf1. Trades between brokers and institutional investors and dealers are likely to imply interest payments on both collateral and short proceeds, but small investors are not likely to earn interest on short-sale proceeds. Further, Regulation T of the Board of Governors of the Federal Reserve Board requires a cash margin of 50% on short positions (Curley [5]). The margin requirement of less than 100%, coupled with the payment of perhaps different interest rates on short sales, complicates a great deal the formal analysis of short sales. With a margin of 100% on short positions and a single interest rate being paid on short-sales deposits, simple re-scaling can take care of the alternative situations regarding whether interest payments based on this single rate are received by the investor or not, according to Levy and Sarnat [11, Chapter 9].

As Alexander notes, the type of short restrictions faced by investors is defined at the individual security level, not at the aggregate portfolio level as was the case with Lintner's definition. Simple re-scaling of weights is not possible. Moreover, investors must decide simultaneously which securities are to be sold short and purchased long as well as the proper investment allocations [2, p. 1502]. The calculation of optimal weights under such conditions is not possible using conventional portfolio optimization. Alexander uses a two-step procedure based on the Elton-Gruber-Padberg (EGP) algorithm. It is appropriate, therefore, to review briefly how the EGP algorithm works.

Portfolio theory assumes investors care only about risk and returns and that historical data on returns suffices to provide the information needed to make allocation decisions among a set of securities. In Markowitz's model [14], descriptive statistics calculated on sample data are used to approximate expected individual security returns. In other formulations (diagonal or single-index model, market model), statistical information on particular securities is calculated on the basis of an index [6 and 17]. Risk and portfolio returns are calculated as a weighted average of individual security statistics. The investor then minimizes the risk of the portfolio subject to a given return.

Elton, Gruber, Padberg [6] seem to have had had two major objectives in mind when they developed their portfolio optimization algorithms. First, they wanted to develop a practical portfolio optimization algorithm. Consequently, they focused on the single-index model, which states that investors can use an index to summarize information on individual securities but which makes claims about equilibrium conditions. The single-index model is also known as

Sharpe's diagonal model; but in EGP's formulation, a single, tangent portfolio is obtained without requiring the investor to set the required return for the optimal portfolio. This is important because it provides a "base case" to work with in the realistic short-sales optimization.

EGP algorithms offer other advantages. For example, they make quadratic programming unnecessary and, more importantly, provide insights into the role of short sales in a single-index based formulation [4].

The major thrust of the EGP algorithm is easy to describe. The investor's security selection problem is similar to what is known in the operations research literature as the "knapsack problem." A traveler must pack a limited number of objects in a knapsack, and the objects have different value in terms of weight and usefulness. The best course of action for this traveler (investor) is to select those objects (securities) that have the greatest usefulness-to-weight (return-to-risk) ratio. It can be shown that the traveler could set a cut-off point for this ratio and accept (reject) those objects with individual ratios above (below) the cut-off point. It is plausible to think this insight may have led EGP to their development of their algorithms, since Padberg is a leading researcher in the mathematics of combinatorial optimization and operations research.

Elton, Gruber, Padberg [6, 7, 8, 9] developed their algorithms for portfolio selection by applying Khun-Tucker's optimization conditions to the following function:

$$\theta = (r_p - r_f)/\ \sigma_p \tag{1}$$

First, securities are ranked in terms of their Treynor index $(r_i\text{-}r_f)/\beta_i$. Then a set of Z_i is calculated, which will determine optimal weights ($w_i^* = Z_i/\Sigma\ Z_i$). The variable Z_i reflects how good each security is compared to the "cut-off" point C*. In the EGP algorithms, securities are included (excluded) in the portfolio according to whether their Treynor indicator is higher (lower) than a critical value C* from the set of C_i variables. The larger the difference between their indicator and the critical value, the more heavily weighted they are in the optimal portfolio, since $w_i^* = Z_i/\Sigma_i\ Z_i$.

$$C_i = \frac{\sigma_m^2 \sum_j \frac{(r_i - r_f)\beta_i}{\sigma_{e_i}^2}}{1 + \sigma_m^2 \sum_j \beta_i / \sigma_{e_i}^2} \tag{2}$$

$$Z_i = \beta_i / \sigma_{e_i^2} [(r_i - r_f) / \beta_i - C^*] \tag{3}$$

In the no-short-sales case, EGP note that C* corresponds to the value for which the Treynor security indices are larger than or equal to C_i. Z_i is positive for those securities and zero otherwise. For the short sales case, the critical value "C*" is the C_i of the lowest ranked security in the sample. The variable "r_f" in the EGP algorithm refers to the risk-free rate, and it corresponds to r_{f1} in our case.

Now, we can return to the realistic short-sales case and to Alexander's application of the EGP algorithm. When a security is sold short, the seller expects a return that is a mixture of the margin-adjusted (m) security return itself, the margin-adjusted interest (r_{f2}) payments on short sales proceeds, and the interest rate (r_{f1}) payments on collateral. Risk must also be adjusted accordingly (see Alexander [2, 3]). Below we offer a succinct comparison of the algebraic differences between the standard and the realistic short-sales cases.

	Standard case	Realistic case
Security return:	$-r_i$	$(-r_i/m)+r_{f1}+(r_{f2}/m)$
Security risk:		
Beta	$-b_i$	$- b_i /m$
Residual Var.	$-\sigma_I^2$	$-\sigma^2_i/(m^2)$

Table 1 presents the "long" and "short" values for single-index data for a ten-security example. The original data is from Elton and Gruber [7] and is also used by Alexander [3] to illustrate his optimization technique.

Alexander [2, 3] first proposes augmenting the set of initial securities with "artificial variables" to reflect the associated margin constraints and interest requirements at the individual security level. Security data used in optimizations has to reflect the different values that returns, betas, and residual variances will have, depending on whether they are held long or short. In Table 2, securities 1 to 10 represent the original or "long" values for the securities under consideration. Securities 11 to 20 represent the artificial securities, and the corresponding transformation rules reflect their possible values when the original securities are sold short.

Table 1
Data for the 10 security example

Stock	r_iL	r_iS	b_iL	b_iS	σ_i^2L	σ_i^2S	TreynorL	TreynorS
1	15	-17	1	-2	50	200	10	11
2	17	-21	1.5	-3	40	160	8	8.66667
3	12	-11	1	-2	20	80	7	8
4	17	-21	2	-4	10	40	6	6.5
5	11	-9	1	-2	40	160	6	4.66667
6	11	-9	1.5	-3	30	120	4	3.5
7	11	-9	2	-4	40	160	3	3.75
8	7	-1	0.8	-1.6	16	64	2.5	3
9	7	-1	1	-2	20	80	2	2.66667
10	5.6	1.8	0.6	-1.2	6	24	1	

where:

r_iL = return from stock i when held long;
r_iS = return from stock i when held short = $(-r_iL/m)+r_{f1}+(r_{f2}/m)$;
b_iL = beta for stock i when held long;
b_iS = beta stock i when sold short = $- b_iL/m$;
σ_i^2L = residual variance for stock i when held long;
σ_i^2S = residual variance for stock i when held short = $\sigma^2_iL/(m^2)$;
TreynorL = (riL -rf1)/biL, when the security is held long;
TreynorS = (riS - rf1)/biS, when the security is held short;
m = 50%, Federal Reserve's margin on short sales;
rf_1 = 5%, risk free rate;
rf_2 = 4%, risk free rate paid to short seller on short sale proceeds;
σ_m^2 = 10, variance of the market rate of return; and
T = 60, number of observations.

Note: Data from Alexander [3], who uses Elton and Gruber's [7] data for the long case.

Alexander then suggests the application of the Elton-Gruber-Padberg "no short-sales" algorithm to the resulting set of real and artificial securities to determine the optimal portfolio weights. In particular, Alexander states that "The Elton, Gruber, and Padberg algorithm can still be used to find the solution, provided that some modifications are made. All that needs to be done is to take the set of N risky securities and artificially create a second set of N securities. ... The EGP no-short-selling algorithm is then applied to the resulting set of 2N securities." [2, p. 1502]. Similar statements can be found in Alexander [3, p. 67] and [2, p. 1504].

Table 2
Original and “artificial” securities data

Stock	r_i	b_i	$\sigma_i^2\sigma$	Treynor	Security Ranking	Treynor
1	15	1	50	10	11	11
2	17	1.5	40	8	1	10
3	12	1	20	7	12	8.66667
4	17	2	10	6	2	8
5	11	1	40	6	13	8
6	11	1.5	30	4	3	7
7	11	2	40	3	15	7
8	7	0.8	16	2.5	14	6.5
9	7	1	20	2	4	6
10	5.6	0.6	6	1	5	6
11	-17	-2	200	11	16	4.66667
12	-21	-3	160	8.66667	6	4
13	-11	-2	80	8	18	3.75
14	-21	-4	40	6.5	17	3.5
15	-9	-2	160	7	7	3
16	-9	-3	120	4.66667	19	3
17	-9	-4	160	3.5	20	2.66667
18	-1	-1.6	64	3.75	8	2.5
19	-1	-2	80	3	9	2
20	1.8	-1.2	24	2.66667	10	1

Note: r_i for security 11 is equal to r_1S in Table 1, and is the return from stock 1 when held short. That is, $r_{11} = r_1S = (-r_1L/m)+rf1+(rf2/m) = (-15/.5)+(5)+(4/0.5) = 17$.

Our initial attempts to replicate Alexander’s results failed. Careful analysis of his research indicated that Alexander [2, 3] might not accurately describe the numerical procedures employed. Alexander’s results can be obtained and presented in a more comprehensive optimization framework.

First, it does not appear that the EGP algorithm in any of its forms can be applied to a set containing both original and artificial securities. The EGP algorithm requires the securities to be ranked in descending order in terms of their Treynor coefficient. The last two columns of Table 1 and the Treynor rankings in Table 2 show that doing so would cluster the first five securities and their artificial counterparts at the top ten positions of the table. Applying EGP procedures to the whole set of 2N securities, as Alexander suggests, would result in optimal weights that select the first five securities twice --which makes no sense.

Second, if a complete set of artificial securities were included in an EGP optimization table, the algorithm would be unable to recognize the fact that the original and "artificial" securities are the same, which would cause the real security and its "image" to be held both long and short simultaneously. Alexander [2] indicates why a portfolio optimization algorithm would never hold the same security long and short, but his general comments are drawn from standard models that do not use artificial securities. Alexander's EGP model needs additional restrictions to make the algorithm aware of the implicit relationship between real and artificial securities.

We obtain Alexander's results using the EGP algorithm by performing the following steps:

Run the EGP procedure to determine which securities will be held long and short. For the example of reference, this has been done by Elton, Gruber and Padberg [7] and is reproduced in Table 3. As noted earlier, the securities are included or excluded in the no-short-sales optimization according to whether they score above a cut-off point. As the calculation in column 7 (Table 3) shows, the cut-off point represents the point at which the Treynor coefficients for each security are above the coefficients in column 6. In the no-short-sales case, the cutoff point is the maximum value (cmax) in column 6 (Table 3), which in our case is the boldfaced number 5.451056. As Cheung and Kwan [4] note, at this cutoff point the correlation of the security groupings and the market portfolio is the highest.

Modify the data for the securities to be shorted as indicated in Table 1. Table 4 presents the results of such modifications. The securities to be held long (1 to 5) are not changed, and the artificial securities for these (11 to 15) are not included. With respect to the securities to be held short, the artificial securities (16 to 20) are included but the original securities (6 to 10) are not.

Run the EGP procedure with standard short-sales restrictions. Table 5 presents the output from such operation. Note the weights obtained are identical to Alexander's (1995) "Restricted Model" [3, p. 67, Exhibit 2, column 5]. Employing the no-short-sales cutoff point (Cmax = 5.451056) the case would replicate the results presented in Table 3 because the top portion of the table remains unchanged after the transformation performed in step 2.

These procedures are different from what Alexander stated in points 1 to 3 above. Note also that negative signs for weights should never be obtained directly from the optimization, since the logic of dealing with realistic short sales conditions requires positive weights. In Alexander's own words "short selling can be thought of as purchasing securities with negative betas and covariances." [3, p. 68; see also the second paragraph on page 67]. Negative signs can be added to help in the reading of the results, but doing so is unnecessary and may be misleading.

Step three is at odds with the following statement by Alexander: "Unfortunately, the EGP algorithm cannot be used to identify the tangency portfolio when unrestricted short selling is allowed. Instead, matrix inversion is necessary" [3, p. 65]. Neither Elton, Gruber, Padberg [6, 8] nor any other researcher such as Kwan [10] identify any problems in applying the EGP algorithm to the standard short-sales case, which permits full use of short-sale proceeds by the investor. Finally, Alexander's transformation involves using negative betas. Alexander [2] mentions that a modification of the EGP algorithm due to Kwan [10] is capable of handling negative betas. As we have shown, Alexander [3] seems to use the traditional EGP algorithm throughout his numerical example, without any further reference to Kwan's method.

In summary, our experience in studying Alexander's procedures to obtain optimal weights via artificial variables [3, p. 67, Exhibit 2, column 5 "restricted case"] yields the following main observations: a) Alexander does not build a complete set of artificial securities, and b) Alexander does not include both the original and artificial securities in the same optimization. These observations could have fatal repercussions on Alexander's methodology since they divorce the economic rationale for his model presented in [2] from the numerical implementation of his algorithm presented in [3]. A well-specified model would carry the economic content of the theory and would not require arbitrary manipulations of the set of securities to be optimized. An example of such model is presented in the next section.

Table 3
EGP algorithm. Discovering securities to hold short and long

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
					C_i	Z_i	W_i
1	0.2	0.02	0.20000	0.20000	1.666667	0.090979	0.23477
2	0.45	0.05625	0.65000	0.07625	3.687943	0.095585	0.246657
3	0.35	0.05	1.00000	0.12625	4.41989	0.077447	0.199851
4	2.4	0.4	3.40000	0.52625	5.429142	0.109789	0.283309
5	0.15	0.025	3.55000	0.55125	**5.451056**	0.013724	0.035414
6	0.3	0.075	3.85000	0.62625	5.301205	0	0
7	0.3	0.1	4.15000	0.72625	5.022693	0	0
8	0.1	0.04	4.25000	0.76625	4.906205	0	0
9	0.1	0.05	4.35000	0.81625	4.747613	0	0
10	0.06	0.06	4.41000	0.87625	4.517286	0	0
						0.387524	1

Where each of the columns represent the Elton-Gruber-Padberg algorithm: (1) Security number; (2) $(r_i\text{-}rf_1)b_i/\sigma_i^2$; (3) b_i^2/σ_i^2; (4) Cumulant of column 2 (from security one to "i"); (5) Cumulant of column 3 (from security one to "i"); (6) $C_i = (\sigma_m^2 * \text{Column 4})/[(1 + \sigma_m^2) * \text{column 5}]$; (7) $Z_i = (bi/\sigma_i^2)((R_i - Rf_1)/bi) - Cmax)$; and (8) $w_i = Z_i/\Sigma Z_i$.

Table 4
Data modifications for Alexander's procedures

Security #	Mean Return	Excess Return	Beta	Unsystematic Risk	Excess Return over Beta
1	15	10	1	50	10
2	17	12	1.5	40	8
3	12	7	1	20	7
4	17	12	2	10	6
5	11	6	1	40	6
16	-9	-14	-3	120	4.66666667
17	-9	-14	-4	160	3.5
18	-1	-6	-1.6	64	3.75
19	-1	-6	-2	80	3
20	1.8	-3.2	-1.2	24	2.66666667

Table 5
Obtaining Alexander's "restricted short sales" results

(1)	(2)	(3)	(4)	(5)	(6) C_i	(7) Z_i	(8) W_i
1	0.2	0.02	0.20000	0.02000	1.666667	0.103508	0.127
2	0.45	0.05625	0.65000	0.07625	3.687943	0.119078	0.146
3	0.35	0.05	1.00000	0.12625	4.41989	0.108771	0.134
4	2.4	0.4	3.40000	0.52625	5.429142	0.235083	0.289
5	0.15	0.025	3.55000	0.55125	5.451056	0.029385	0.036
16	0.35	0.075	3.90000	0.62625	5.370025	0.003948	0.005
17	0.35	0.1	4.25000	0.72625	5.143722	0.033115	0.041
18	0.15	0.04	4.40000	0.76625	5.079365	0.026865	0.033
19	0.15	0.05	4.55000	0.81625	4.965894	0.045615	0.056
20	0.16	0.06	4.71000	0.87625	4.824584	0.107896	0.133
			30.61000	5.11125		0.813263	1

Note: Columns (1) to (6), and (7) are calculated as in Table 3. However, column (7) uses C(last)=4.824584, as in EGP's short sales case.

III. A MIXED-INTEGER MODEL

The previous section noted that portfolio selection resembled a "knapsack" problem. Suppose now that the objects to be selected by the traveler may turn out to have different value once he or she is on the road. There will not be a second chance to pack. For example, a thermos filled with hot chocolate may have different usefulness on a sunny day than on a cold weather day. The cautious traveler would mentally calculate the different carrying values of each object under alternative scenarios. The final packing, therefore, would include those objects with the highest carrying value under a set of conditions. (Alternatively, one may think about carrying different knapsacks subject to a maximum number or maximum weight constraint.)

The investor's problem may be understood in a similar manner. The securities selected may have different values depending on whether they are bought or sold short and on whether the market goes up or down. At the overall portfolio level, it can be shown that short sales may improve the risk-adjusted return profile of the portfolio even in growing stock market conditions. In general, the portfolio selection problem is to find the best packing of securities under a variety of economic scenarios.

The realistic short-sales optimization could be formulated as a quadratic, mixed-integer problem, as follows.

Select w_i, δ_i to

$$\text{Max} \quad (r_p - r_{f1})/(\sigma_p) \tag{4}$$

where

$$r_p = \Sigma_i w_i [r_i L + \delta_i (r_i S - r_i L)] \tag{5}$$

$$\begin{aligned} \sigma_{p2} &= [(\Sigma_i w_i b_i)^2 \sigma_{m2}] + (\Sigma_i w_i^2 \sigma_i^2)] \\ &= \Sigma_i w_i^2 (b_i L + \delta_i (b_i S - b_i L))^2 \sigma_{m2} \\ &\quad + \Sigma_i w_i^2 (\sigma_i^2 L + \delta_i (\sigma_i^2 S - \sigma_i^2 L)) \end{aligned} \tag{6}$$

subject to

$$\delta_i \text{ is binary, either 0 or 1.}$$
$$w_i \geq 0,\ i=1, ..., n.$$
$$\Sigma w_i = 1$$

The "switching " or binary variable δ_i selects the corresponding short or long values for the variables of interest (r_i, b_i, and σ_i). A value of zero (one) indicates the security will be held long (short) as in, for example, $r_i = r_i L + \delta_i (r_i S - r_i L)$. The algorithm selects the best "packs" of securities. Further fine tuning of the proportions (w_i's) in each of these packs provides the overall optimal portfolio.

It is appropriate to provide further detail on the characteristics of the model presented in this section. Mixed-integer optimizations are rather common in operations research. The mixed-integer model in (4) can be solved via enumeration, but simple branch-and-bound algorithms can solve the problem more efficiently. Such algorithms perform a standard search over the binary tree determined by the δi's, whose nodes correspond to subproblems obtained by restricting some variables to the values of zero or one and whose nodes are programmed to search for the upper planes of this binary tree or polytope. In the mixed-integer algorithm, quadratic programming is used to eliminate the spherical excess, given that optima for these types of functions are upper planes on the polytope having a quadratic residual. Finally, the use of integer programming in the context of our problem constitutes a relaxation strategy to deal with the complexity (combinatorial nature) of the problem. Further detail on the switching binary variable or mixed-integer programming approach can be found at Tarrazo [19].

Table 6
Switching variable model: results

Stock	W_i^*	δ_i^*	b_i	RESVAR	R_i	W_i^{**}
1	0.127205	0	0.127205	0.80905	1.908068	0.127
2	0.146245	0	0.219368	0.855504	2.486166	0.146
3	0.133807	0	0.133807	0.358086	1.605684	0.134
4	0.289032	0	0.578065	0.835397	4.913549	0.289
5	0.036186	0	0.036186	0.052387	0.398048	0.036
6	0.004828	1	-0.01448	0.0022797	-0.04345	-0.005
7	0.040658	1	-0.16263	0.264491	-0.36592	-0.041
8	0.033558	1	-0.05369	0.072074	-0.03356	-0.034
9	0.056032	1	-0.11206	0.251164	-0.05603	-0.056
10	0.132449	1	-0.15894	0.421026	0.238408	-0.132
	1	5	0.592819	3.921967	11.05096	1
				3.514342	(systematic risk)	
				7.436309	(total risk, variance)	

Portfolio data:

Return	11.05096
Beta	0.592819
Standard deviation	2.72696
Return to variability	2.21894
Number secs. long	5
Number secs. short	5
Percentage of wealth long	73.25%
Percentage of wealth short	26.75%
Correlation with the market	0.687453

wi* are the optimal weights, wi** are the same weights but rounded to three significant decimals and with a negative sign added, as in Alexander's [1995].

Table 6 shows the results from optimizing the model presented for the data in Table 1. The results are identical to those obtained by Alexander, but our model determined the securities to be held long and short after a selective and effective evaluation of possible combinations. No arbitrary exclusions of variables were needed.

Alexander [2] explains how the investor would compare the short and long values for each security and never buy and short the same security simultaneously --also known as a short sale against the box. However, his algorithm never offers this choice to the investor, since some of Alexander's original and artificial securities must be excluded from the optimization for the algorithm to work. In sum, the mixed-integer variable model supports Alexander's [2] economic rationale of the restricted short-sales methodology through a more comprehensive and rigorous numerical analysis framework.

In spite of its sophistication, the mixed-integer problem can be solved with EXCEL's Solver by changing some of the default optimization settings (time, iterations, precision, tolerance, estimates, derivatives, and search).[1]

IV. CONCLUDING COMMENTS

The contributions of Alexander [2, 3] are path-breaking in several critical areas. First, they enhance the range of outcomes the investor must consider before purchasing securities. Second, they open the door to understanding the role of cash and short sales in portfolio selection, which has been eluding researchers for over 40 years. Third, the ingenuity of applying trading restrictions at the security level enables these procedures to enhance investment practice. In addition, Alexander's procedures are compatible with techniques to cope with estimation risk. The mixed-integer model presented here is a natural way to model the dual nature of investments data and, therefore, may help to improve the determinations of security weights under realistic short sales restrictions in uncertain economic environments.

NOTES

1. Please contact the authors to obtain a copy of the spreadsheets (EXCEL 5.0 for Windows 3.1) used in this study.

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