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Yes, CAPM Is Dead

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ABSTRACT

This paper proves CAPM, as derived by Sharpe, Lintner, and Mossin, faces either a serious problem of endogeneity or of circularity, or both. CAPM thus cannot serve the purposes originally intended for it. The expectation of an unconditional random variable is a constant parameter as set by the density function and should not be affected by other factors. However, CAPM asserts the expected excess rate of return on an asset depends on its beta, which in turn depends on the covariance between the asset's return and the market portfolio's rate of return. In effect, the expected rate of return for an asset must be assumed or known to form the covariance matrix necessary for CAPM. It follows that beta depends on the expected rate of return for an asset, not vice versa, exactly opposite of the presumed relationship. This paper also addresses the validity of the market index model used in empirical studies.

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I. INTRODUCTION AND BACKGROUND OF CAPM

Sharpe (1964), Lintner (1965), and Mossin (1966) developed the capital asset pricing model (CAPM) to explain the relationship between the expected rate of return and the risk of capital assets. CAPM now serves as the foundation of modern financial economics. However, what theoretical problems lie at the base CAPM? Would the applications and related findings still be valid or reliable?

The empirical verification of CAPM has been extensively discussed in the literature, though Fama and French take the lead. For example, Fama and French (1992) claim "CAPM is useless for precisely what it was developed to do." Based on their empirical results, Berg (1992) cites Fama's claim "beta as the sole variable explaining returns on stocks is dead." Fama and French (1996) continue their challenge in "The CAPM is wanted, dead or alive."

Fama and French (2004) suggest that problems with CAPM may be attributed to the simplifying assumptions and the difficulties in using the market portfolio in the tests of the model. Indeed, given the difficulty of formulating a good proxy for the market portfolio, the consensus is to reject CAPM as signifying the sole source of risk, and resort to what appear to be ad–hoc factor models. Any remaining disagreement, as noted in Fama and French (2004), initially appears to focus on the source of those other factors. However, the "bad" news doesn't stop. Harvey, Liu, and Zhu (2014) provide an extensive review of the literature about factor–models and suggest "that most claimed research findings in financial economics are likely false," not at all a narrow claim.

The predominant attacks on CAPM have been empirical in nature, leaving the underlying theory virtually unscathed. The complexity of the logic of the scientific method as applied within the social sciences makes it especially difficult to analyze the theoretical foundation of CAPM. However, if the underlying theory itself is flawed or problematic in ways not previously analyzed, then the conclusions based on the theory may not be true regardless of the care taken in empirical testing. This paper aims to examine the logical status of CAPM, and suggests the futility of continued use of CAPM, with the caveat that no alternatives appear in sight. In this respect, we agree with Fama and French (1992) and others about the death of CAPM, though we offer a different approach in arriving at this conclusion.[1] Our paper differs from previous studies by considering the possibility that CAPM has severe limits even as a theoretical model.

Our paper is organized as follows. Section I provides the background of the problem presented. Section II explores the meaning and definition of the "market portfolio" in CAPM. Section III delves into an analysis of risk return relationship within CAPM. Section IV shows the market risk premium in CAPM is decided by the expected excess rate of return on assets and the invertible covariance matrix among the assets' rate of return. Section V proves algebraically that CAPM is not a legitimate empirical asset-pricing model, because the product of the beta and risk premium is rewritten based on the expected excess rate of return. Section VI explores statistical problems of the market index model in empirical studies. Section VII presents the significance of our results.

II. WHAT IS THE MARKET PORTFOLIO IN CAPM?

Assume there are n risky securities with one risk–free asset. The notation $\tilde{R}_t$ is the stochastic rate of return on risky asset i and its expected rate of return is R_i, for i=1, 2, …, n. The risk–free rate of return on the risk–free asset is r, an n x 1 vector of the expected excess rates of return is R - r, and the n x n non–singular covariance matrix of risky assets' rates of return is Ω. The non–singular covariance matrix Ω and the n x 1 vector R - r imply there exists a unique n x 1 vector λ, such that the expected excess rate of return vector R - r can be expressed as:[2]

$$R - r = \Omega \lambda = c_\omega \Omega \omega \tag{1}$$

where $\lambda = \Omega^{-1}(R-r)$, yielding $\lambda = c_\omega \omega$, with ω being an n x 1 vector, such that $\omega = \Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$, $c_\omega = e^T\Omega^{-1}(R-r) = \omega^T(R-r)/\omega^T\Omega\omega$ is a constant scalar, and e^T is a 1 x n vector of ones. Since $e^T \omega = 1$, ω is interpreted as a portfolio in this paper.

With a risk–free asset, the optimal risky portfolio for individual investors within the mean–variance framework can be derived by maximizing the Sharpe Ratio subject to the constraint of total weights equal to one [3]. The first order condition, which is the necessary and sufficient condition for the optimal portfolio solution, is: [4]

$$(R-r) - e^T\Omega^{-1}(R-r)\, \Omega\omega^* = 0 \tag{1'}$$

The solution of the first order condition for the optimal portfolio selection ω* must uniquely equal $\omega^* = \Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$, which is exactly the risky mutual fund with a risk–free asset in the two–fund separation theorem. The optimal portfolio ω* (a decision variable) depends on the parameter, not vice versa, as shown in Equation (1).

Since Equation (1') is the necessary and sufficient condition for the optimal portfolio solution ω* within the mean–variance framework, and Equation (1') mathematically equals (1), the unique ω in Equation (1) must be the optimal mean–variance efficient portfolio ω*. The following proposition presents this result.

Proposition 1. Given the expected excess rate of return vector R - r on n risky securities and the non–singular covariance matrix Ω between n risky securities rate of returns, the portfolio ω in Equation (1) is the unique risky optimal mean–variance efficient within the mean–variance framework if and only if $\omega = \Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$.

Although individual investors may have different weights for the risk–free asset and the risky fund (the optimal risky portfolio) ω*, the composition of the optimal portfolio, as shown in Equation (1) and two–fund separation theorem, must be the same for all investors. Consequently, the composition of the aggregated optimal risky portfolio for all investors in the market must be ω*, the optimal portfolio herein.
Equation (2) represents the typical form for CAPM:

$$R_i = r + \beta_i (R_m - r), \quad \text{for all } i=1, 2, \ldots, n \tag{2}$$

where $R_m - r$ is the market risk premium, $\beta_i = Cov(\tilde{R}_m, \tilde{R}_i)/\sigma_m^2$, Cov(.,.) is the covariance operator, ω_m is the market portfolio, R_m is the expected rate of return on the market portfolio, with $\tilde{R}_m = \omega_m^T \tilde{R}$, and σ_m^2 is the variance of the market portfolio rate of return.

From Equation (2), $\beta_i = E[(\tilde{R}_m - R_m)(\tilde{R}_i - R_i)]/\sigma_m^2$, where, E[..] is the expectation operator. The definition of β_i implies that an asset's beta depends on the expected rates of return, R_i on asset i and R_m on the market portfolio, i.e., in a normal statistical setting, the requisite expected rates of return must be exogenous variables (inputs). This fact creates important implications for understanding the theoretical foundation and empirical application of CAPM. Simply stated, standard reasoning for CAPM suggests the expected return for an asset is a function of the asset's beta as based upon systematic risk, yet an asset's beta is also a function of its expected return. This is either a serious problem with endogeneity in empirical work, or a straightforward problem of circular reasoning from the standpoint of logic, or both. We contribute to the debate about CAPM by focusing on this broad puzzle within its foundations.

To follow through and identify the problem in more detail, consider the underlying math. In terms of vector algebra, Equation (2) can be rewritten as:

$$R - r = \beta(R_m - r) = (R_m - r)\Omega\omega_m/\sigma_m^2 = c_m\Omega\omega_m = \Omega\lambda_{m}, \tag{3}$$

where, , $\beta = \Omega\omega_m/\sigma_m^2$, $c_m = (R_m - r)/\sigma_m^2$, $\lambda_m = c_m\omega_m$, β is the n x 1 vector of beta, $\Omega\omega_m$ is the n x 1 vector of covariances between the rate of return on i[th] risky asset $\tilde{R}_m$ $i = 1,2,\ldots,n$ and the market portfolio rate of return $\tilde{R}_m$, $\sigma_m^2 = \omega_m^T\Omega\omega_m$.

Equations (1) and (3) imply that $\lambda = \Omega^{-1}(R-r) = \lambda_m$ and thus the market portfolio ω_m in Equation (3) must be identical to the unique portfolio $\Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)] = \omega$ in Equation (1).[5] That is, Equation (3) must be identical to Equation (1). Since equation (1) holds for any positive integer n, Equation (3) must hold for any number of assets as well once the expected rate of return and their covariance existent or being given.

Equation (1) is an algebraic result and is irrelevant to the demand and supply of risky assets. Therefore, Equation (1) is irrelevant to market equilibrium. With the assumption that Ω is invertible, Equation (3) follows from Equation (1). If Ω is irrelevant to market equilibrium, then Equation (3) is also irrelevant to market equilibrium. Should this be the case, market equilibrium is irrelevant to CAPM, and the required rate of return on an asset based on the market equilibrium in CAPM must equal the assumed expected rate of return on an asset in Equation (1).

In addition, the object of Equation (3) is the optimal mean–variance efficient portfolio for n securities rather than the market equilibrium. In other words, previous studies substitute the market portfolio under the equilibrium condition in CAPM for the unobservable mean–variance efficient portfolio. This substitution is not justifiable. Unfortunately, Elton et al. (2010) and Fama and French (2004) argue that if all investors select the same optimal portfolio, then, in equilibrium, the portfolio must be a portfolio in which all securities are held in the same percentage that they represent in the market value. This could overstate the role played by the market portfolio in CAPM because, as shown in Equations (1) and (3), the relationship of $R - r = \beta(R_m - r)$ always holds for any n assets once the portfolio is constructed by these n assets, and is satisfied by $\omega_m = \Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$. Therefore, the exact market portfolio can be re–defined as all securities in the market weighted by their ex–post market–value, or by a finite number of securities with the optimal mean–variance efficient as presented by ω in Equation (1).

Similarly, as shown in Equations (1) and (3), assumptions about: (1) investor's risk preference or utility function, (2) unlimited risk–free rate lending and borrowing, (3) the need for a joint normal density function, and (4) a perfect market are not needed to derive CAPM.

Since the ex–ante expected rates of return are exogenous to CAPM, there is no rationale for using the ex–post systematic risk in Equations (2) and (3) to explain the already existent ex–ante expected excess rate of return. Therefore, should CAPM be valid, the expected excess rate of return would have depended on the portfolio ω_m, which affects the beta or systematic risk and depends on the expected excess rate of return. This leads a mathematical problem that the expected excess rate of return on an individual security is a function of the expected rate of return on the market portfolio, which depends on the expected excess rate of return on the asset. In other words, the expected excess rate of return on securities R–r depend on itself; the expected excess rate of return on securities. Thus, CAPM suffers from a serious problem of endogeneity or circularity, or both; and even appears to be a tautology. Equation (3) proves the following proposition.

Proposition 2. Given the expected excess rate of return vector R–r and non–singular covariance matrix Ω between the risky securities returns, CAPM holds if and only if the market portfolio $\omega_m = \Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$ [6].

The implication of Propositions 1 and 2 is that if the market portfolio is not the optimal mean–variance efficient portfolio, then Equation (1) and CAPM fail, and CAPM is invalid. In other words, CAPM entails that the market portfolio is the optimal mean–variance efficient portfolio. Thus, to test CAPM is equivalent to testing the optimal mean variance efficiency of the market portfolio ω_m. Since the components of means and the covariance between n risky securities' returns are unobservable, the optimal mean–variance efficiency of market portfolio ω_m must be unobservable as well. As shown in Proposition 2, the optimal mean–variance portfolio ω_m is irrelevant to the market and thus is not a real market portfolio. However, to facilitate the following exploration of CAPM, the ω_m in CAPM refers to the market portfolio for the remainder of this paper.

Since the portfolio ω_m in Equation (3) depends on the expected excess rate of return on assets and the covariance among the assets' returns, the weights of the portfolio ω_m must be changed if randomly appearing information (e.g., earning surprise or other events) causes the changes of the parameters such as expected rate of returns or the covariance on the assets. Furthermore, Equation (1) fails for all portfolios on the mean variance efficient frontier except the optimal one. That is, the ω_m in CAPM is not just the mean variance efficiency but also the optimal one.

Proposition 2 shows that the market portfolio ω_m in CAPM should only be restricted to its components and its composition must be decided by $\Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-$

r)]. In other words, if the assets are excluded in the construction of ω_m, CAPM (or Equation (2) or (3)) should not apply to these risky assets even though these assets are in the market. For example, consider an asset with ex–ante positive expected excess rate of return. The expected excess return on this asset calculated by CAPM should be zero if this asset is absent in the composition of the portfolio ω_m and its rate of return is uncorrelated to the rate of return on the portfolio ω_m. This example demonstrates that CAPM is misspecified when it is applied to assets excluded from the construction of the portfolio ω_m in Equation (3).

The unobservable nature of the market portfolio should be attributed to the unobservable population parameters of the expected excess rate of return R–r and the covariance in Ω for these n risky securities rather than all risky assets in the market. Thus, the claim that the market portfolio in CAPM includes the non–tradable and human capital assets is not sustainable from the construction of ω_m as shown in the Proposition 2.

III. IN CAPM, THE GREATER EXPECTED RATE OF RETURN ON AN ASSET, THE GREATER THE BETA

Both the beta and the market risk premium are vital factors for the expected excess rate of return in CAPM. Beta is the core variable in CAPM that affects the expected excess rate of return on the assets. In CAPM, only the asset's non–diversifiable systematic risk merits a reward for the asset's expected excess rate of return. The beta represents systematic risk and is defined as the ratio of the covariance between the market rate of return and the security rate of return to the variance of market rate of return. From Proposition 2, the market portfolio ω_m equals $\Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$, thus the beta in CAPM can be rewritten in terms of the algebraic vector:

$$\beta = \Omega\omega_m/\sigma_\omega^2 = \Omega\Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)\sigma_\omega^2] = (R-r)/(c_m\sigma_\omega^2). \qquad (4)$$

Since $c_m\sigma_\omega^2$ is a constant, Equation (4) demonstrates β in CAPM depends on and is proportional to the expected excess rate of return on assets, rather than an independent variable denoting systematic risk that is related to the covariance between the rate of return on assets and the return on the market portfolio. The greater expected rate of return results in a greater beta, which runs contrary to the assertion of CAPM that a higher beta should have a higher expected return to compensate investors' risk taking.

Furthermore, as shown in the last term of Equation (4), beta as the measure of the systematic risk disappears because β is irrelevant to the covariance between the market portfolio rate of return and the asset's rate of return. In other words, the beta in CAPM is completely irrelevant to systematic risk, if the market portfolio is the optimal mean–variance efficient as shown in Equation (3). That is, there is no such thing as systematic risk in CAPM if the market portfolio $\omega_m = \Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$. Therefore, the claim that the discount rate on a project in corporate finance is a function of the project's beta is not sustainable on the basis of Equation (4).

Unfortunately, beta has been misinterpreted as the systematic risk in CAPM for the last four decades. Equation (4) proves the following Proposition:

Proposition 3. In CAPM, if the market portfolio ω_m equals $\Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$, then beta depends on the expected excess rate of return, not vice versa. The greater the expected return the greater the beta, rather than the greater beta the higher the expected rate of return.

IV. THE MARKET RISK PREMIUM IS IRRELEVANT TO INVESTORS' RISK PREFERENCE

The other vital factor for the R – r in CAPM is the market risk premium $R_m - r$, which is defined as the product of the market portfolio ω^T_m and the vector of expected excess rate of return R – r. Thus, the market risk premium $R_m - r$ can be presented as

$$R_m - r = \omega_m^T (R-r) = (R-r)^T \Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)] \qquad (5)$$

Equation (5) shows the market risk premium as computed by ex–ante population parameters; the expected rates of return and the covariances among the stochastic rate of returns.[7] Once the expected excess rate of return and the covariance in the Ω are given, the market risk premium $R_m - r$ is set and fixed. The risk preference of investors in the market plays no role in determining the market risk premium. Equation (5) shows the following proposition.

Proposition 4. In CAPM, the market risk premium $R_m - r$ is solely determined by the parameters of the expected excess rates of returns and the non–singular covariance matrix between the risky asset returns if the market portfolio is equal to $\Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$ rather than by the risk preferences of investors in the market.

V. THE TROUBLESOME LOGICAL FOUNDATION OF CAPM

In statistics, if the rate of return on an asset is not conditional on the market rate of return at the outset of developing a model, the ex–ante expected rate of return on an asset in CAPM must be a constant and should be decided by its density function only.[8] Hence the ex–ante constant expected rate of return on an asset should not depend on other parameters or variables such as the systematic risk or the market risk premium.

Smith and Walsh (2013) argue that CAPM is a tautology. We tend to be partial to such a claim and are tempted to suggest that the following proof demonstrates the tautological nature of CAPM. With the assumption that the covariance matrix Ω is invertible, the vector of the expected excess rate of return R–r in CAPM can be rewritten as

$$R - r = \Omega\Omega^{-1}(R-r) = [e^T\Omega^{-1}(R-r)]\Omega\Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$$

$$= [e^T\Omega^{-1}(R-r)]\, \Omega\omega_m = c_m\Omega\omega_m = (R_m-r)\Omega\omega_m / \sigma_m^2 = \beta(R_m-r), \qquad (6)$$

where as defined in Equation (2), $\omega_m = \Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$, $\beta = \Omega\omega_m/\sigma_m^2$, and $c_m = (R_m-r)/\sigma_m^2 = e^T\Omega^{-1}(R-r)$.

Equation (6) shows that the constant expected excess rate of return R–r on an asset can be rewritten in different forms such as $\Omega\Omega^{-1}(R - r)$ and $\beta(R_m - r)$, which is CAPM. Since $\beta(R_m - r)$ is just a substitution for the expected excess rate of return on assets, R - r, it appears that CAPM is a tautology and mathematically invalid as an asset pricing model. Equation (6) proves the following proposition:

Proposition 5: Given the expected excess rate of return vector R–r and non–singular covariance matrix Ω between the risky securities returns, CAPM is a tautology if and only if the market portfolio $\omega_m = \Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$. If the market portfolio $\omega_m \neq \Omega^{-1}(R-r)/[e^T\Omega^{-1}(R-r)]$, then the linear relationship between the expected rate of return on an asset and its beta in CAPM fails and CAPM is invalid.[9]

Since Equation (6) holds for any finite n securities, n=1,2,3,..,n, it should also hold for any subset of capital market securities, given the expected return and invertible covariance between all other securities returns in the subset.

In practice, the proxy for the market portfolio is the market index, which is constructed with different sizes and weights at the time of the index being created. Regardless of its weight and/or size, the observable market indices are fixed per se and independent of the expected excess rate of return on assets and the covariance between n assets' returns in the future. This implies that CAPM fails as shown in Proposition 5, if the market index does not exactly equal the optimal mean–variance efficient portfolio ω_m.

In other words, if the market index is not the optimal mean–variance efficient portfolio and is used to substitute for the optimal mean–variance market portfolio ω_m, the linearity between the expected excess rate of return and the beta calculated by this market index is not sustainable as argued by Roll and Ross (1994). This could also be one of the reasons why such poor coefficients of determination R^2 occur when testing CAPM, in that researchers use a mean–variance inefficient market index instead of the optimal mean–variance efficient market portfolio. The other reason for poor empirical results in testing CAPM could be applying CAPM to assets which are excluded in the construction of the market portfolio (or market index).

A numerical example can clearly demonstrate these results. Assume n=2, and the vector of the ex–ante expected excess rate of return and the covariance matrix are given, respectively, by the following: [10]

$$(R\text{-}r) = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \quad \Omega = \begin{pmatrix} 4 & 3 \\ 3 & 9 \end{pmatrix}.$$

The inverse matrix Ω^{-1} can be calculated as:

$$\Omega^{-1} = \frac{1}{27}\begin{pmatrix} 9 & -3 \\ -3 & 4 \end{pmatrix}.$$

A constant of $c_m = e^T\Omega^{-1}(R-r)$ can be calculated as:

$$c_m = (1 \quad 1)\begin{pmatrix} 9/27 & -3/27 \\ -3/27 & 4/27 \end{pmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \frac{40}{27}$$

From the investor's optimal portfolio selection within the mean-variance framework, as shown in Equation (1'), the optimal solution of ω^* from solving the first order condition $(R–r) – e^T\Omega^{–1}(R–r)\ \Omega\omega^* = 0$ is $\omega^* = \Omega^{–1}(R–r)/[e^T\Omega^{–1}(R–r)]$, and ω^* can be calculated as:

$$\begin{bmatrix} \omega_1^* \\ \omega_2^* \end{bmatrix} = \frac{1}{27}\begin{pmatrix} 9 & -3 \\ -3 & 4 \end{pmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix}\frac{27}{40} = \begin{bmatrix} 15 \\ 25 \end{bmatrix}\frac{1}{40} = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

Since the investor's optimal portfolio (presents the investors' demand for the assets) must equal to the market portfolio ω_m, which represents the supply, to obtain the equilibrium for CAPM, the optimal portfolio ω^* derived from the first order condition here must be the market portfolio ω^*. Hence, as shown in Equation (1), the $c_\omega = e^T\Omega^{–1}(R–r) = \omega^{*T}(R–r)/\omega^{*T}\Omega\omega^* = (R_m – r)/\sigma_m^2$, thus the first order condition of optimal portfolio for maximizing the Sharpe ratio for investors is exactly CAPM.

However, λ in Equation (1) is calculated by:

$$\lambda = \Omega^{–1}(R–r) = \frac{1}{27}\begin{pmatrix} 9 & -3 \\ -3 & 4 \end{pmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 15 \\ 25 \end{bmatrix},$$

and the unique portfolio vector ω in Equation (1), is determined by:

$$\omega = \omega_m = \Omega^{–1}(R\text{-}r)/[e^T\Omega^{–1}(R\text{-}r)] = \frac{1}{27}\begin{bmatrix} 15 \\ 25 \end{bmatrix}/(40/27) = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

Here, ω_m is totally irrelevant to the market value of the assets. The market risk premium $R_m - r$, the variance of the market portfolio rate of return s_m^2, and beta are calculated, respectively, by:

$$R_m\text{-}r = \omega_m^T\ (R\text{-}r) = (3/8 \quad 5/8)\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \frac{65}{8}$$

$$\sigma_m^2 = \omega_m^T\Omega\omega_m = (3/8 \quad 5/8)\begin{pmatrix} 4 & 3 \\ 3 & 9 \end{pmatrix}\begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix} = (3/8 \quad 5/8)\begin{bmatrix} 27/8 \\ 54/8 \end{bmatrix} = \frac{351}{64}$$

$$\beta = \Omega\omega_m/\sigma_m^2 = \begin{bmatrix} 27/8 \\ 54/8 \end{bmatrix}/(351/64) = \begin{bmatrix} 8/13 \\ 16/13 \end{bmatrix}$$

With two risky assets, n=2, CAPM is presented by:

$$\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \text{R-r} = \Omega\Omega^{-1}(\text{R-r}) = [e^T\Omega^{-1}(\text{R-r})]\ \Omega\Omega^{-1}(\text{R-r})/\ [e^T\Omega^{-1}(\text{R-r})]$$

$$= c_m\Omega\omega_m = (R_m\text{-r})\ \Omega\omega_m/\sigma_m^2 = \beta(R_m\text{-r}) = \begin{bmatrix} 8/13 \\ 16/13 \end{bmatrix} \frac{65}{8}$$

This numerical example shows: (i) the given constant expected excess rate of return can be rewritten in different forms, including the product of the beta β and the market risk premium $R_m - r$, (ii) given the covariance matrix Ω, the optimal mean–variance market portfolio ω_m, beta, and market risk premium are all constructed and calculated by the given expected excess rate of return, (iii) CAPM is just an algebraic result once the expected excess rate of return and the non–singular covariance matrix are given, and (iv) CAPM is a tautology if the optimal mean variance efficient portfolio is used in CAPM.

Under the same expected rate of return and the covariance matrix, if the market index is value–weighted by ω_x^T = (1/3, 2/3), then the market risk premium $R_x - r$, variance σ_x^2, and beta β_x under this market index are calculated, respectively, by:

$$R_x\text{-r} = \omega_x^T\ (\text{R-r}) = \begin{pmatrix} 1/3 & 2/3 \end{pmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \frac{25}{3}$$

$$\sigma_x^2 = \omega_x^T\Omega\omega_x = \begin{pmatrix} 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 9 \end{pmatrix} \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{pmatrix} 1/3 & 2/3 \end{pmatrix} \begin{bmatrix} 10/3 \\ 21/3 \end{bmatrix} = \frac{52}{9}$$

$$\beta_x = \Omega\omega_x/\sigma_x^2 = \begin{bmatrix} 10/3 \\ 21/3 \end{bmatrix} /(52/9) = \begin{bmatrix} 30/52 \\ 63/52 \end{bmatrix}$$

CAPM with the mean–variance inefficient market index is:

$$\beta_x(R_x - r) = \begin{bmatrix} 30/52 \\ 63/52 \end{bmatrix} \frac{25}{3} = \begin{bmatrix} 250/52 \\ 525/52 \end{bmatrix} \neq \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \text{R - r}$$

This demonstrates that if the optimal mean–variance efficient market portfolio is substituted for a non–optimal mean–variance efficient index regardless how it was weighted, the required rate of return will be miscalculated by the product of the beta and the market risk premium in CAPM. Thus CAPM fails if the market index is not exactly the optimal mean–variance efficient as stated in proposition 5.

However, despite our numerical examples and the intuition underlying these examples, we refrain from agreeing with Smith and Walsh's (2013) conclusion that CAPM is a tautology, for one basic reason. Namely, the presumed proof that CAPM is a mathematical (or logical) tautology depends upon assuming that the covariance matrix Ω is invertible. Matrices are not invertible by definition (whether mathematical or logical). Indeed the assumption that Ω is invertible may function in a manner similar to the claim that an asset's risk depends upon market clearing, and is thus a substantive assumption. Consequently, our primary conclusion is that CAPM has a serious problem with endogeneity, in that an asset's beta depends upon the expected rates of return of all assets in the market portfolio, yet that asset is part of the market portfolio.

It might be countered that the equilibrium nature of the original proofs for CAPM circumvent the problem of endogeneity, because beta and expected returns are determined simultaneously. Once all relevant information is processed by the market, equilibrium generates market risk and returns. But accepting this view entails that CAPM is not testable, because any test of the truth of CAPM must make the assumption that the market is in equilibrium, and presumably whether or not the market is in equilibrium is an empirical question. Hence we conclude that either CAPM has a serious problem with endogeneity or with circularity. We elaborate on this conclusion by considering the empirical nature of CAPM.

VI. IS THE MARKET INDEX A PROPER EXPLANATORY VARIABLE?

Debate about the testability of CAPM should end. In empirical studies of CAPM, the proxy for the market portfolio, which does not depend on the mean and the covariance between the securities' rate of return, has been used as a substitute for the mean–variance market portfolio. Although the weights of the market index are fixed and that index is not the mean–variance efficient portfolio[11], the rate of return on the market index is used as the explanatory variable to account for the rate of return on the security (or the mutual fund) in empirical studies.

Is the market index a proper explanatory variable? This section addresses the validity of the market index model, which has been used as the basic model by previous empirical studies. In the regression model, the rate of the return on the market index may still be a valid explanatory variable if the return on the market index is not a function of the dependent variable of the rate of return on the risky asset. The following single market index model has been used in the empirical studies in finance.
The single market index model is:

$$\tilde{R}_{i,t} - r = \alpha_i + \beta_i(\tilde{R}_{m,t} - r) + \tilde{\varepsilon}_{i,t} \tag{7}$$

where the $\tilde{R}_{i,t}$ is the security rate of return at time t, $\tilde{R}_{m,t}$ is the market index rate of return at time t, $\tilde{\varepsilon}_{i,t}$ is the error term at time t. α_i is the intercept term. The S&P 500 market index has been used by most researchers as the proxy for the market portfolio in finance literature. Based on Equation (7), the total risk of the security return can be decomposed into the systematic and unsystematic risk for return on the security if the error term $\tilde{\varepsilon}_{i,t}$ is independent of the return on the market index $\tilde{R}_{m,t}$. Systematic risk is measured by the $\beta_i{}^2\sigma_m{}^2$, while the unsystematic (or firm–specific) risk is the variance of the $\tilde{\varepsilon}_i$.

The estimated alpha α_i in Equation (7) has been used to measure the risk–adjusted abnormal rate of return in the empirical studies under the condition that CAPM is valid. If CAPM is a valid model, the expectation of the estimated alpha α_i for i^{th} asset in Equation (7) should be zero. If the estimated alpha is significantly different from zero, one can conclude the existence of the anomalies and/or the market is inefficient. However, as shown in Proposition 5 and in the numerical example in the last section, since the market index is not exactly the mean variance efficient market portfolio almost surely, CAPM is invalid when using the market index. The invalidity of CAPM could cause significant non–zero alpha findings. Thus, the conclusions of anomalies and/or market inefficiency based upon the significant non–zero estimated alpha could be misleading and are clearly questionable.

In reality, the market index is composed of a finite number of individual securities in the market. These securities' returns must be given before the market index return can be calculated and determined in practice. As a result, the return on the market index is affected by the return on these securities and not vice versa. However, Equation (7) shows that the rate of return on the market index affects the rate of return on the individual security. This is not consistent with the real world and the assumption of dependency between the excess return on a security and the return on the market index underlying the market model.

Furthermore, if the rate of return, $\tilde{R}_{i,t}$ on i^{th} security, is also a component of the market index return $\tilde{R}_{m,t}$, then $\tilde{R}_{i,t}$ is not just the dependent variable but also one of components of the market index, the explanatory variable. This implies that the regression model of Equation (7) is misspecified. If such a misspecified model is accepted, then why would another index like the return on the industry index not be used to substitute for the return on the market index? After all, the industrial index would be better than the market index for predicting or describing the rate of return on the security because it is more highly correlated to its own industry index return than the return on the market index in the real world. Taken to the extreme, though it is meaningless, the individual security rate of return (i.e., the tautology model) is a perfect explanatory variable to describe its own rate of return. In particular, if Equation (7) is used as the basic model to measure the performance of mutual funds in empirical studies, the misspecification of Equation (7) would be even more significant, because mutual funds would be components of the market index as well.

Another problem for Equation (7) is that the error term $\tilde{\varepsilon}_{i,t}$ which is the systematic risk for asset i, is not independent of the explanatory variable of the return on market index $\tilde{R}_{m,t}$ if $\tilde{R}_{i,t}$ is one of the components of the market index. The non–zero correlation between the error term and the explanatory variable violates the independence assumption between the explanatory variable and the error term in the regression model. In addition, multiplying the weight used to compute the market index to both sides of Equation (7) and then summation over all assets in the index results in $\sum \omega_i \tilde{\varepsilon}_{i,t} + \sum \omega_i \alpha_i = 0$, for all time t. A stochastic term plus a non–stochastic term equal to zero implies that $\sum \omega_i \alpha_i = 0 = \sum \omega_i \tilde{\varepsilon}_{i,t}$. That is, the error term of i^{th} asset is not independent of other assets' error terms for all time t.

VII. CONCLUSION

This paper uses algebraic analysis to prove that "CAPM is dead," because it either is beset with a serious endogeneity problem or is circular. Given the expected excess return vector, whether or not the market is in equilibrium, the non–singular covariance matrix implies that there must exist one and only one portfolio, such that the expected excess rate of return on assets can be rewritten as the product of its beta and the market risk premium as presented in Equation (6). Since, Equation (1) is the necessary and sufficient condition for solving for the optimal portfolio within the mean–variance framework and CAPM exactly satisfies this relationship, this leads to the mathematical conclusion that the market portfolio in CAPM must be the optimal mean–variance efficient portfolio and must depend on the expected excess return. The optimal mean–variance efficient market portfolio is a necessary and sufficient condition for CAPM. This proposition entails the problem of endogeneity.

Endogeneity by itself may not prove the death of CAPM, but does point to the well–known difficulties with obtaining reliable betas or costs of capital. That is, CAPM appears to be almost useless for predicting the rate of return for an asset in the real world, as claimed by Levi and Welch (2014). Even if endogeneity problems can be resolved, the fundamental argument for CAPM appears circular, which alone is an obstacle that appears insurmountable. Our argument here is not empirical in nature. Rather it is based on the logic and mathematics of CAPM, which distinguishes our argument from the many other empirical or econometric critiques of CAPM.

ENDNOTES

1. For example, the size effect is explored by Banz (1981), Keim (1983), Roll (1981), Reinganum (1982), Chan and Chen (1991), Chan et al. (1985), and others. The factors of the market to book value and earnings/price ratio on the excess return are examined by Fama and French (1992, 1993, 1996, 2004), Chan et al. (1991), and Bansal et al. (2005). The zero beta is studied by Shanken (1985) and Roll (1985).
2. Bold face fonts denote vectors or matrices. The i^{th} element of R–r is R_i–r; the expected rate of return on the asset i minus risk–free rate r. It is not the intent of this paper to estimate the expected excess rate of return R–r or the elements of Ω. See Theorem1, pp. 15 by Ichiro (1975).
3. The optimal risky portfolio with risk–free asset is the tangent point on the efficient frontier from the risk–free rate.
4. The first order condition $(R-r) - [\omega^T(R-r)/(\omega^T\Omega\omega)]\Omega\omega = 0$ is used to solve the optimal portfolio decision ω rather than to derive the constant parameter of expected excess rate of return R-r in CAPM. Multiplying $e^T\ \Omega^{-1}$ in both sides of this first order condition and given the total weighs of portfolio $e^T\omega = 1$ results in $[\omega^T(R-r)/(\omega^T\Omega\omega)] = e^T\ \Omega^{-1}(R-r)$ and Equation (1'). See Equation (7), pp. 596 by Lintner (1965), or Equation 6.1(or 13.1), pp.102 (or 291) by Elton et al. (2010). More precisely, Equation (1) is the necessary and sufficient condition for maximizing the Sharpe ratio.
5. $\lambda = \Omega^{-1}(R-r) = \lambda_m$ or $\lambda = c\omega = c_m\omega_m = \lambda_m$ implies $ce^T\omega = c_m e^T\omega_m$, or $c=c_m$. Therefore $c(\omega-\omega_m) = 0$ or, $\omega = \omega_m$.

6. Roll (1977) reaches an "if and only if" relationship between return/beta and the market portfolio mean–variance efficiency. The orthogonal portfolio z to the market portfolio in Roll's paper plays the same role of risk–free in this paper (see Corollary 6, pp. 165).
7. Pablo and Del Campo Baonza (2010) report that the average MRP used in 2010 by professors in the USA is 6.0%.
8. The Arbitrage Pricing Theory (APT) developed by Ross (1976) assumes the rate of return on a security is generated by multiple common factors, whereas the expected rate of return and the covariance in CAPM are assumed being given at the beginning of the model setting.
9. The non–singular matrix Ω is a one–to–one and onto isomorphic mapping, different portfolios will be mapped onto different expected excess rate of returns and vice versa by Ω.
10. Without loss of generality, all percentages in the rate of returns 5% and 10% and the standard deviations 20% and 30% are omitted. In this example, the correlation coefficient in this example is assumed = 0.5. Both of the expected excess rate of return and covariance are assumed in Sharpe, Lintner, and Mossin derivation.
11. For detailed versions of Equations 3–10, see Greene (1997), pp. 64.

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