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Optimal Portfolio of Corporate Investment and Consumption under Market Closure

Detao Zhang[a] and Tian Zhang[b*]
[a] *School of Economics, Shandong University*
27 Shanda Nanlu, Jinan 250100, P. R. China
zhangdetao@gmail.com
[b] *Department of Mathematics, University of Southern California*
3620 S. Vermont Avenue, Los Angeles, CA, 90089, USA
tianzhan@usc.edu

ABSTRACT

In this paper, we present the model of corporate optimal investment with consideration of the difference between the day time and night time. In the model, the investor has three market activities of his choice: corporate investment, savings in a bank, and consumption. The optimal strategies for the investor are obtained using the Hamilton-Jacobi-Bellman equation which is derived using the dynamic programming principle. After introducing the model in general, a specific case, the *Hyperbolic Absolute Risk Aversion* case, will be discussed in details, where the explicit optimal strategy can be obtained using a very simple and direct method. At the very end, we present some simulation results along with a brief analysis of the relationship between the optimal strategy and other factors.

JEL Classifications: *G11, G12*

Keywords: optimal investment; consumption choice; corporate project; market closure

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I. INTRODUCTION

Portfolio diversification plays an important role in finance. In Merton (1987) and Duffie (1992), a stochastic model was first used to analyze the security market, while Karatzas (1987), based on the stochastic analysis, presented the model considering the split of consumption and investment of the investors. Choi (1989), on the other hand, paid attention to the international corporate investment where investments were made into real corporate projects. Afterwards, Bellalah and Wu (2008) extended the theory of corporate international investment from Choi into an environment with the presence of incomplete information and taxation from governments. Their model was concerned about the international diversification problem in finance and gave analysis to the “home bias puzzle”. In 2006, a model giving the optimal corporate portfolio and consumption choice was presented by Wu and Zhang (2006), while Bellalah and Wu (2002) modeled the market closure together with the international securities portfolio management with incomplete information.

In our paper, we present a model of corporate investment and consumption choice problem with market closure that we can derive the optimal investment strategy. Our model is an extension of the usual corporate investment model from Wu and Bellalah (2002), (2008), Wu and Zhang (2008), Huang and Wu (2006), since we take consideration of the investment differentiation during the day and time.

In the model, we suppose that the investor has the option to invest his money into a bank account to accrue interest rate, where the interest rate can follow any interest rate model such as the CIR model or the Vasicek model. The wealth in the bank account will follow the basic exponential format.

On the other hand, the investor can invest his money into a real corporate production project as well. The investor might choose the corporate project to get higher return to his investment while higher risk will be brought at the same time. In real life, production of the factory is different in the day and night; therefore, in our model, we have equations describing the input price and the output price of the production in the day time and night with specified instantaneous expected rates and instantaneous volatilities. Also, we developed two pieces of production equations corresponding to the day-time and night-time production which is a function of the output price. With the input and output price model and the production quantity model, we can easily derive the expression for the net profit of the corporate investment after considering taxation of the government.

Finally, instead of making investment, the investor has the option to make consumption for satisfaction as well. During the night time, banking activities will not be allowed; therefore the investment portfolio will be the same throughout the night. However, the investor is still allowed to consume to increase utility. Clearly, the consumption in the day time and the night will be different.

In general, we consider corporate investment, non-risky bond investment, and consumption altogether to maximize the utility of wealth for the investor aiming to find out the optimal strategy for the investment. For the general model, it is difficult to solve for the optimal value function explicitly, therefore we considered a specific and important case of utility function-*HARA (Hyperbolic Absolute Risk Aversio*n) case to get the optimal solution. It turns out that we can write out the explicit optimal solution using a very simple and direct method which will be discussed in Section II of the paper.

In the next section, we will present the model and the assumptions of the optimal investment problem. In the HARA cases, besides giving the explicit optimal solution of the corporate investment model using dynamic programming principle, we will give analysis of the model in the economical point of view in Section III.

At the end, some simulation results will be given to provide a more numerical interpretation of the model. Relationship between the optimal portfolio and important factors such as the volatility and the interest rate will be given together with computer-drawn graphs.

II. MODEL AND FORMULATION OF PROBLEM

Let (Ω, F, P) be a complete probability space endowed with a filtration $\{F_t : 0 \le t \le T\}$. $\{B_t\}_{0 \le t \le T}$ is 1-dimensional Brownian Motion defined in this space. We suppose that the investor can put his money in a bank account to get non-risky reward following the equation

$$dP_0(t) = r(t)P_0(t)dt \quad (1)$$

where $r(t)$ is the interest rate in terms of time t.

On the other hand, investor can choose a real corporate project with some production to get a higher return but with certain risk. The input price and output price are not the same during the day and the night. The duration of the production in the day time is T, and the duration at the night time is N.

The input price P(t) and the output price S(t) of the production during the day are described as

$$dP(t) = \alpha_p(t)P(t)dt + \sigma(t)P(t)dB(t) \quad (2)$$

and

$$dS(t) = \alpha_s(t)S(t)dt + \sigma(t)P(t)dB(t)$$
$$t \in [0, T], [T+N, 2T+N], \ldots\ldots [(n-1)(T+N), nT+(n-1)N] \quad (3)$$

The input price P(t) and the output price S(t) of the production during the night are described as

$$dP(t) = \bar{\alpha}_p(t)P(t)dt + \bar{\sigma}(t)P(t)dB(t) \quad (4)$$

And

$$dS(t) = \bar{\alpha}_s(t)S(t)dt + \bar{\sigma}(t)S(t)dB(t)$$
$$t \in [T, T+N], [2T+N, 2T+2N], \ldots\ldots [nT+(n-1)N, n(T+N)] \quad (5)$$

where P(t) and S(t) have initial values P_0, S_0, respectively. Here α_p, $\bar{\alpha}_p$ represent the instantaneous expected input rates in the day and the night, respectively, while α_s, $\bar{\alpha}_s$ represent the instantaneous expected output rates. Finally σ, $\bar{\sigma}$ are instantaneous volatilities in the day and the night, respectively.

The quantity of the production in the day is

$$Q(t) = [S(t)]^{\beta_1},\ t \in [0,T], [T+N, 2T+N], \ldots\ldots [(n-1)(T+N), nT+(n-1)N] \tag{6}$$

and at night

$$Q(t) = [S(t)]^{\beta_2},\ t \in [T, T+N], [2T+N, 2T+2N], \ldots\ldots [nT+(n-1)N, n(T+N)] \tag{7}$$

In economic theory, β_1 and β_2 are negative constants in general. The cash flow from this project is

$$R(t) = (1-\tau)[S(t) - P(t)]Q(t) \tag{8}$$

where P(t) and S(t) are input price and the output price; Q(t) is the quantity of the output and τ is the tax rate. Using this expression, it is easy to get the following results. In the day, $t \in [0,T], [T+N, 2T+N], \ldots\ldots [(n-1)(T+N), nT+(n-1)N]$

$$P(t) = P((k-1)(T+N))e^{\int_{(k-1)(T+N)}^{t}(\alpha_p(r) - \frac{1}{2}\sigma^2(r))dr + \int_{(k-1)(T+N)}^{t}\sigma(r)dB_r} \tag{9}$$

and

$$S(t) = S((k-1)(T+N))e^{\int_{(k-1)(T+N)}^{t}(\alpha_S(r) - \frac{1}{2}\sigma^2(r))dr + \int_{(k-1)(T+N)}^{t}\sigma(r)dB_r} \tag{10}$$

Then,

$$Q(t) = [S(t)]^{\beta_1} = S^{\beta_1}((k-1)(T+N))e^{\beta_1\int_{(k-1)(T+N)}^{t}(\alpha_S(r) - \frac{1}{2}\sigma^2(r))dr + \beta_1\int_{(k-1)(T+N)}^{t}\sigma(r)dB_r} \tag{11}$$

and

$$R(t) = (1-\tau)S^{\beta_1}((k-1)(T+N))e^{(1+\beta_1)\int_{(k-1)(T+N)}^{t}\sigma(r)dB_r}F(t) \tag{12}$$

where

$$F(t) = \left\{ S((k-1)(T+N))e^{\int_{(k-1)(T+N)}^{t}(\alpha_S(r) - \frac{1}{2}\sigma^2(r))dr} - P((k-1)(T+N))e^{\int_{(k-1)(T+N)}^{t}(\alpha_p(r) - \frac{1}{2}\sigma^2(r))dr} \right\} e^{\beta_1\int_{(k-1)(T+N)}^{t}(\alpha_s(r) - \frac{1}{2}\sigma^2(r))dr} \tag{13}$$

At night, $t \in [T, T+N], [2T+N, 2T+2N], \ldots\ldots [nT+(n-1)N, n(T+N)]$

$$P(t) = P(kT+(k-1)N)e^{\int_{kT+(k-1)N}^{t}(\bar{\alpha}_p(r) - \frac{1}{2}\bar{\sigma}^2(r))dr + \int_{kT+(k-1)N}^{t}\bar{\sigma}(r)dB_r} \tag{14}$$

and

$$S(t) = S(kT+(k-1)N)e^{\int_{kT+(k-1)N}^{t}(\bar{\alpha}_s(r)-\frac{1}{2}\bar{\sigma}^2(r))dr+\int_{kT+(k-1)N}^{t}\bar{\sigma}(r)dB_r} \tag{15}$$

Then,

$$Q(t) = [S(t)]^{\beta_2} = S^{\beta_2}(kT+(k-1)N)e^{\beta_2\int_{kT+(k-1)N}^{t}(\bar{\alpha}_s(r)-\frac{1}{2}\bar{\sigma}^2(r))dr+\beta_2\int_{kT+(k-1)N}^{t}\bar{\sigma}(r)dB_r} \tag{16}$$

and

$$R(t) = (1-\tau)S^{\beta_2}(kT+(k-1)N)e^{(1+\beta_2)\int_{kT+(k-1)N}^{t}\bar{\sigma}(r)dB_r}\,\bar{F}(t) \tag{17}$$

where

$$\bar{F}(t) = \left\{S(kT+(k-1)N)e^{\int_{kT+(k-1)N}^{t}(\bar{\alpha}_s(r)-\frac{1}{2}\bar{\sigma}^2(r))dr} - P(kT+(k-1)N)e^{\int_{kT+(k-1)N}^{t}(\bar{\alpha}_p(r)-\frac{1}{2}\bar{\sigma}^2(r))dr}\right\}e^{\beta_2\int_{kT+(k-1)N}^{t}(\bar{\alpha}_s(r)-\frac{1}{2}\bar{\sigma}^2(r))dr} \tag{18}$$

Applying Ito's formula to R(t), we can get, when $[(k-1)(T+N), kT+(k-1)N]$,

$$dR(t) = R(t)f(t)dt + R(t)(1+\beta_1)\sigma(t)dB_t \tag{19}$$

where $f(t) = \frac{1}{2}(1+\beta_1)^2\sigma^2(t) + \frac{F'(t)}{F(t)}$, $F(t) \neq 0$, and the input price is not equal to the output price. And

$$dR(t) = R(t)\bar{f}(t)dt + R(t)(1+\beta_2)\bar{\sigma}(t)dB_t,\ t \in [kT+(k-1)N, k(T+N)] \tag{20}$$

where $\bar{f}(t) = \frac{1}{2}(1+\beta_2)^2\bar{\sigma}^2(t) + \frac{\bar{F}'(t)}{\bar{F}(t)}$ and $\bar{F}(t) \neq 0$.

Let X(t) denote the total wealth at time t and $\pi(t)$ represent the proportion of the wealth invested in the real project, then $(1-\pi(t))X(t)$ is the amount invested in non-risky bond.

$$\begin{aligned} dX(t) = {} & [r(t)X(t) - C_1(t) + (f(t)-r(t))\pi(t)X(t)]dt \\ & + (1+\beta_1)\sigma(t)\pi(t)X(t)dB_t,\ t \in [(k-1)(T+N), kT+(k-1)N] \end{aligned} \tag{21}$$

and

$$\begin{aligned} dX(t) = {} & [r(t)X(t) - C_2(t) + (\bar{f}(t)-r(t))\pi(t)X(t)]dt \\ & + (1+\beta_2)\bar{\sigma}(t)\pi(t)X(t)dB_t,\ t \in [kT+(k-1)N, k(T+N)] \end{aligned} \tag{22}$$

In the day time, the investor can choose investment with proportion π and consumption rate C_1 to maximize his wealth, but at the night time, he cannot change his portfolio and only can choose a different consumption rate C_2, the portfolio at the night time stays same as the optimal portfolio in the day time. So the investor wants to maximize the following utility of wealth by choosing his investment strategy π and consumption rate C_1 and C_2. Let $J^1(X)$ be the value of wealth X starting the day time and $J^2(X)$ be the value of the wealth X starting the night time. The whole investment duration can be divided into n days, i.e., nT day time duration and nN the night time duration.

$$J^1(X_0) = \max_{(\pi,C_1)} E[\int_0^T e^{-\gamma t} U(C_1(t))dt + e^{-\gamma t} J^2(X_T)] \tag{23}$$

$$J^2(X_T) = \max_{C_2} E_T[\int_T^{T+N} e^{-\gamma t(t-T)} U(C_2(t))dt + e^{-\gamma N} J^1(X_{T+N})] \tag{24}$$

$$J^1(X_{(n-1)(T+N)}) = \max_{(\pi,C_1)} E_{(n-1)(T+N)}[\int_{(n-1)(T+N)}^{nT+(n-1)N} e^{-\gamma(t-_{(n-1)(T+N)})} U(C_1(t))dt + e^{-\gamma T} J^2(X_{nT+(n-1)N})] \tag{25}$$

$$J^2(X_{nT+(n+1)N}) = \max_{C_2} E_{nT+(n-1)N}[\int_{nT+(n+1)N}^{n(T+N)} e^{-\gamma(t-nT+(n-1)N)} U(C_2(t))dt + e^{-\gamma N} h(X_{n(T+N)})] \tag{26}$$

This is one kind of stochastic optimal control problem which can be solved by using the celebrated dynamic programming principle.

$$\begin{cases} \frac{\partial \bar{J}^2}{\partial t} + \max_{C_2}\left\{ \frac{\partial \bar{J}^2}{\partial t}\left[r(t)X(t) - C_2(t) + (\bar{f}(t) - r(t))\pi(t)X(t) \right] + e^{-\gamma t}U(C_2(t)) \right. \\ \left. + \frac{1}{2}\frac{\partial^2 \bar{J}^2}{\partial X^2}(1+\beta_2)^2 \bar{\sigma}^2(t)\pi^2(t)X^2(t) \right\} = 0, t \in \left[nT + (n-1)N, n(T+N) \right], \\ \bar{J}^2(X, n(T+N)) = h(X); \end{cases} \tag{27}$$

$$\begin{cases} \frac{\partial \bar{J}^1}{\partial t} + \max_{(C_1,\pi)}\left\{ \frac{\partial \bar{J}^1}{\partial t}\left[r(t)X(t) - C_1(t) + (f(t) - r(t))\pi(t)X(t) \right] + e^{-\gamma t}U(C_1(t)) \right. \\ \left. + \frac{1}{2}\frac{\partial^2 \bar{J}^1}{\partial X^2}(1+\beta_1)^2 \sigma^2(t)\pi^2(t)X^2(t) \right\} = 0, t \in \left[(n-1)(T+N), nT + (n-1)N \right], \\ \bar{J}^1(X, nT+(n-1)N) = \bar{J}^2(X, nT+(n-1)N); \end{cases} \tag{28}$$

$$\begin{cases} \frac{\partial \bar{J}^2}{\partial t} + \max_{C_2}\left\{ \frac{\partial \bar{J}^2}{\partial t}\left[r(t)X(t) - C_2(t) + (\bar{f}(t) - r(t))\pi(t)X(t) \right] + e^{-\gamma t}U(C_2(t)) \right. \\ \left. + \frac{1}{2}\frac{\partial^2 \bar{J}^2}{\partial X^2}(1+\beta_1)^2 \bar{\sigma}^2(t)\pi^2(t)X^2(t) \right\} = 0, t \in \left[T, T+N \right], \\ \bar{J}^2(X, T+N) = \bar{J}^1(X, T+N); \end{cases} \tag{29}$$

and

$$\begin{cases} \frac{\partial \bar{J}^1}{\partial t} + \max_{(C_1,\pi)} \left\{ \frac{\partial \bar{J}^1}{\partial t} \left[r(t)X(t) - C_1(t) + (f(t) - r(t))\pi(t)X(t) \right] + e^{-\gamma t} U(C_1(t)) \right. \\ \qquad \left. + \frac{1}{2} \frac{\partial^2 \bar{J}^1}{\partial X^2} (1+\beta_1)^2 \sigma^2(t) \pi^2(t) X^2(t) \right\} = 0, t \in [0,T], \\ \qquad \bar{J}^1(X,T) = \bar{J}^2(X,T); \end{cases} \tag{30}$$

where

$$\begin{aligned} &\bar{J}^1(X,(n-1)(T+N)) = J^1(X_{(n-1)(T+N)}) \\ &\bar{J}^2(X, nT + (n-1)N) = J^1(X_{nT+(n-1)N}), \\ &\qquad \ldots\ldots\ldots\ldots, \\ &\bar{J}^2(X,T) = \bar{J}^2(X_T), \bar{J}^1(X,0) = \bar{J}^1(X_0) \end{aligned} \tag{31}$$

Theoretically we can solve the above 2n partial differential equation step by step from backward to get the optimal (π, C_1, C_2) and the optimal value function $J^1(X_0)$. But in fact, it is difficult to get the explicit solution through these partial differential equations. In the next section, we will study one important case of utility function-HARA (Hyperbolic Absolute Risk Aversion) case to get optimal decision and consumption rates explicitly by using a very simple and direct method.

III. HARA CASE: OPTIMAL SOLUTIONS AND ECONOMIC ANALYSIS

In this section, we consider one kind of special case called Hyperbolic Absolute Risk Aversion (HARA) case. For simplicity, we only consider the case where the entire time interval is T+N, T is the duration of the day time, N is the duration of the night time. For general case, the whole interval is T+N, we can get the explicit result by repeating the procedure with the same method.

Let

$$J^1(X_0) = \max_{(C_1,\pi)} E\left[\int_0^T L e^{-\gamma t} \frac{C_1^{1-R}}{1-R} dt + e^{-\gamma t} J^2(X_T) \right] \tag{32}$$

$$J^2(X_T) = \max_{C_2} E\left[\int_T^{T+N} L e^{-\gamma t} \frac{C_2^{1-R}}{1-R} dt + e^{-\gamma t} K \frac{X_{T+N}^{1-R}}{1-R} \right] \tag{33}$$

here γ and R are constants, where $\gamma > 0$, $R \in (0,1)$. We try to get explicit optimal decision π, consumption rates C_1, C_2 and the optimal value function in this case.

Theorem 3.1 Under all the above assumptions, the optimal strategies to the optimal portfolio choice problem (21), (22), (32), (33) for the specific HARA case is given by

$$\begin{cases} J^1(X_0) = \frac{1}{1-R} X_0^{1-R} P_0, \\ J^2(X_T) = \frac{1}{1-R} X_T^{1-R} P_T, \\ \pi^*(t) = \frac{f(t)-r(t)}{R(1+\beta_1)^2 \sigma^2(t)}, t \in [0, T+N], \\ (C_1(t))^* = (1-R)^{-\frac{1}{R}} (P_t)^{-\frac{1}{R}} X_t, t \in [0, T], \\ (C_2(t))^* = (1-R)^{-\frac{1}{R}} (P_t)^{-\frac{1}{R}} X_t, t \in [T, T+N], \end{cases} \tag{34}$$

where P_t satisfies ODEs (37) and (47).

Proof. We first consider the optimal problem (33) at night time. During the night, the investor cannot change his portfolio. His consumption C_2 is the only activity that varies. So the wealth equation in the duration [T,T+N] is:

$$\begin{aligned} dX(t) = & \left[r(t)X(t) - C_2(t) + (\bar{f}(t) - r(t))\pi^*(t)X(t) \right] dt \\ & + (1+\beta_2)\bar{\sigma}(t)\pi^*(t)X(t)dB(t), t \in [T, T+N] \end{aligned} \tag{35}$$

Here π^* is the optimal portfolio in the day time, we will solve it afterward.

We let P_t be one nonnegative deterministic continuous function whose dynamic will be given later. Applying Ito's formula to $\frac{e^{-\gamma(t-T)}}{1-R} X_t^{1-R} P_t$ from T to T+N, we have

$$\begin{aligned} E_T\left[\frac{e^{-\gamma N}}{1-R} X_{T+N}^{1-R} P_{T+N}\right] = & \frac{1}{1-R} X_T^{1-R} P_T + E_T \int_T^{T+N} \Bigg\{ P_t \Bigg[-\gamma \frac{e^{-\gamma(t-T)}}{1-R} X_t^{1-R} \\ & + \frac{e^{-\gamma(t-T)}}{1-R} X_t^{-R} (1-R) \left[r(t)X(t) - C_2(t) + (\bar{f}(t) - r(t))\pi^*(t)X(t) \right] \\ & - \frac{1}{2} R e^{-\gamma(t-T)} X_t^{-1-R} (1+\beta_2)\bar{\sigma}^2(t)(\pi^*(t))^2 X^2(t) \Bigg] \\ & + \frac{e^{-\gamma(t-T)}}{1-R} X_t^{1-R} P_t \Bigg\} dt. \end{aligned} \tag{36}$$

So we can write

$$J^2(X_T) = \frac{1}{1-R} X_T^{1-R} P_T + I + II + III,$$

where

$$I = \max_{C_2} E_T \int_T^{T+N} \frac{e^{-\gamma(t-T)}}{1-R} \left[C_2^{1-R} - (1-R) X_t^{-R} P_t C_2 - R X_t^{1-R} P_t^{1-\frac{1}{R}} \right] dt;$$

$$II = \max_{C_2} E_T \int_T^{T+N} \frac{e^{-\gamma(t-T)}}{1-R} X_t^{1-R} \Big\{ P_t + P_t \Big[-\gamma + r(t)(t-R) + (\bar{f}(t) - r(t))\pi^*(t) - \frac{1}{2} R(1-R)(1+\beta_2)^2 \bar{\sigma}^2(t)(\pi^*(t))^2 \Big] + RP_t^{1-\frac{1}{R}} \Big\} dt;$$

$$III = \max_{C_2} E_T \left[\frac{e^{-\gamma N}}{1-R} X_{T+N}^{1-R} (K - P_{t+N}) \right].$$

Now we let P_t be nonnegative and the solution of the following ordinary differential equation:

$$\begin{cases} -P'_t = \overline{M} P_t + RP_t^{1-\frac{1}{R}}, t \in [T, T+N], \\ P_{T+N} = K. \end{cases} \quad (37)$$

Here

$$\overline{M} = -\gamma + r(t)(1+R) + (\bar{f}(t) - r(t))\pi^*(t) - \frac{1}{2} R(1-R)(1-\beta_2)^2 \bar{\sigma}^2(t)(\pi^*(t))^2. \quad (38)$$

So

$$P_t = e^{\overline{M}(T+N-t)} \left[K + \int_t^{T+N} e^{-\frac{1}{R}\overline{M}(T+N-s)} ds \right]^R, t \in [T, T+N], \quad (39)$$

$$P_T = e^{\overline{M}N} \left[K + \int_t^{T+N} e^{-\frac{1}{R}\overline{M}(T+N-s)} ds \right]^R. \quad (40)$$

We can easily get II=0, III=0. If we take

$$(C_2(t))^* = (1-R)^{-\frac{1}{R}} (P_t)^{-\frac{1}{R}} X_t, t \in [T, T+N], \quad (41)$$

then we can check that I attains its maximum at point C_2^* and I=0. So

$$J^2(X_T) = \frac{1}{1-R} X_T^{1-R} P_T. \quad (42)$$

And then applying Ito's formula to $\frac{e^{-\gamma t}}{1-R} X_t^{1-R} P_t$ from 0 to T, and taking expectation on both sides, we have

$$E\left[\frac{e^{-\gamma T}}{1-R}X_T^{1-R}P_T\right] = \begin{aligned}[t] &\frac{1}{1-R}X_0^{1-R}P_0 + E\int_T^{T+N}\left\{P_t\left[-\gamma\frac{e^{-\gamma T}}{1-R}X_t^{1-R}\right.\right. \\ &+\frac{e^{-\gamma T}}{1-R}X_t^{1-R}(1-R)\left[r(t)X(t)-C_1(t)+(f(t)-r(t))\pi(t)X(t)\right] \\ &\left.-\frac{1}{2}Re^{-\gamma t}X_t^{-1-R}(1+\beta_1)^2\sigma^2(t)\pi^2(t)X^2(t)\right] \\ &\left.+\frac{e^{-\gamma T}}{1-R}X_t^{1-R}P'_t\right\}dt. \end{aligned} \tag{43}$$

So we can write

$$J^1(X_0) = \begin{aligned}[t] &\frac{1}{1-R}X_0^{1-R}P_0 + \max_{(\pi,C_1)} E\int_0^T \frac{e^{-\gamma t}}{1-R}\left[C_1^{1-R} - (1-R)X_t^{-R}P_tC_1\right] \\ &+e^{-\gamma t}X_t^{1-R}P_t\left[(f(t)-r(t))\pi(t) - \frac{1}{2}R(1+\beta_1)^2\sigma^2(t)\pi^2(t)\right] \\ &\left.+\frac{e^{-\gamma t}}{1-R}X_t^{1-R}(P'_t - \gamma P_t + r(t)(1-R)P_t)\right\} \\ &= \frac{1}{1-R}X_0^{1-R}P_0 + IV + V + VI. \end{aligned} \tag{44}$$

where

$$IV = \max_{(\pi,C_1)} E\int_0^T \frac{e^{-\gamma t}}{1-R}\left[C_1^{1-R} - (1-R)X_t^{-R}P_tC_1 - RX_t^{1-R}p_t^{1-\frac{1}{R}}\right]dt;$$

$$V = \max_{(\pi,C_1)} E\int_0^T e^{-\gamma t}X_t^{1-R}P_tL(\pi(t))dt,$$

and

$$L(\pi(t)) = (f(t)-r(t))\pi(t) - \frac{1}{2}R(1+\beta_1)^2\sigma^2(t)\pi^2(t) - \frac{(f(t)-r(t))^2}{2R(1+\beta_1)^2\sigma^2(t)},$$

$$VI = \max_{(\pi,C_1)} E\int_0^T \frac{e^{-\gamma t}}{1-R}X_1^{1-R}\left\{P_t' - \gamma P_t + (1-R)P_t\left[r(t) + \frac{(f(t)-r(t))^2}{2R(1+\beta_1)^2\sigma^2(t)}\right] + RP_t^{1-\frac{1}{R}}\right\}dt, \tag{45}$$

If we take

$$\pi^*(t) = \frac{f(t)-r(t)}{R(1+\beta_1)^2\sigma^2(t)}, \tag{46}$$

it is easy to check that $L'(\pi^*(t)) = 0$, and $L''(\pi^*(t)) < 0$ .Thus the function $L(\pi)$ attains its maximum at point π^* and $L(\pi^*) = 0, V = 0$. Now we let P_t be the solution of the following ODE

$$\begin{cases} -P_t' = MP_t + RP_t^{1-\frac{1}{R}}, t \in [0,T], \\ P_T = e^{\overline{M}N}\left[K + \int_T^{T+N} e^{-\frac{1}{R}\overline{M}(T+N-s)} ds\right]^R. \end{cases} \quad (47)$$

Here

$$M = -\gamma + r(t))(1-R) + \frac{(f(t)-r(t))^2}{2R(1+\beta_1)^2\sigma^2(t)}(1-R). \quad (48)$$

So

$$P_t = e^{M(T-t)}\left[P_T^{\frac{1}{R}} + \int_t^T e^{-\frac{1}{R}M(T-s)} ds\right]^R, t \in [0,T], \quad (49)$$

$$P_0 = e^{MT}\left[P_T^{\frac{1}{R}} + \int_0^T e^{-\frac{1}{R}M(T-s)} ds\right]^R, \quad (50)$$

Then if we take

$$(C_1(t))^* = (1-R)^{-\frac{1}{R}}(p_t)^{-\frac{1}{R}} X_t, t \in [0,T]. \quad (51)$$

One can check that IV attains its maximum at point C_1^*, IV=0. Then the optimal value function is

$$J^1(X_0) = \frac{1}{1-R} X_0^{1-R} P_0. \quad (52)$$

Remark: In HARA case, we can get from simple calculus that the Pratt-Arrow measure of relative aversion $A = R$, $R \in (0,1)$. So the constant R can indicate the investor's attitude to the risk in the investment.

Once knowing the amount of our wealth, we can make a decision on the strategy of the investment according to formula (34). Now let us give the economic analysis for the optimal investment portfolio.

The optimal portfolio π^* can be referred as the speculative demand, which depends on the measure of relative risk aversion R and is influenced by the volatility of the portfolio. π^* will decrease when $\sigma^2(t)$ increases, that is, higher volatility in the corporate market will stimulate investor to put more money in the risk-less investment.

The value of $f(t)-r(t)$ indicates whether the expected reward rate of the corporate project is higher than the interest rate in the bank. π^* will increase when $f(t)-r(t)$ increases, that is, higher difference between the expected reward rate of the corporate project and the interest rate in the bank will stimulate investors to devote more capital in the corporate project.

IV. SOME SIMULATION RESULTS

From the history of price data in the market, we can use statistical method to estimate the parameters in the model. Now let us give a simulating example. In this example, let the coefficients be constants for simplicity, and we only consider the choice at initial time $t=0$.

Example: We take the following parameters depending on the situation of the real market. Choose $S_0=4$, $P_0=3$, $\beta_1=-0.4$, $\beta_2=-0.3$, $\alpha_s=0.15$, $\overline{\alpha}_s=0.1$, $\alpha_p=0.13$, $\overline{\alpha}_p=0.09$, $\gamma=0.6$, $T=2$, $K=5.5$, $R=0.5$.

From the formula (3.3), we can get the conclusion that the optimal portfolio π decreases when the volatility parameter and the interest rate *r* increases. Let us fix one of σ and r and give the relationship between π and each of them.

In Figure 1 and Figure 2, the curve goes just as we have expected. For fixed r=0.04 (see Figure 1), we can get the relationship between the optimal proportion π^* and

Figure 1
Relationship between the volatility of the production σ and the optimal proportion π^*

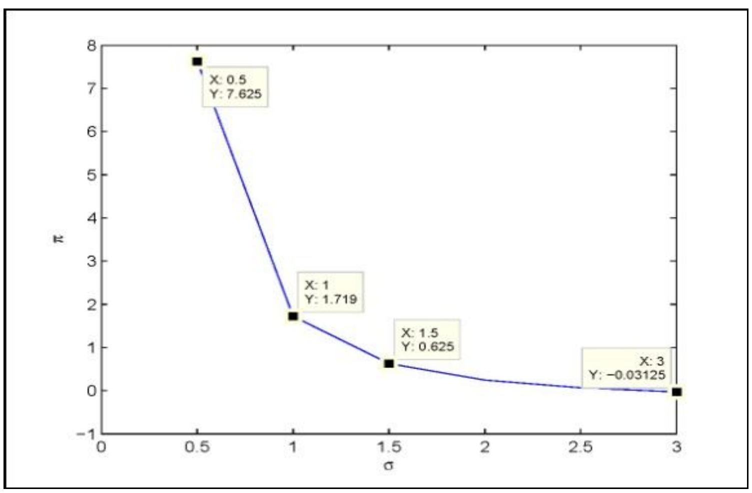
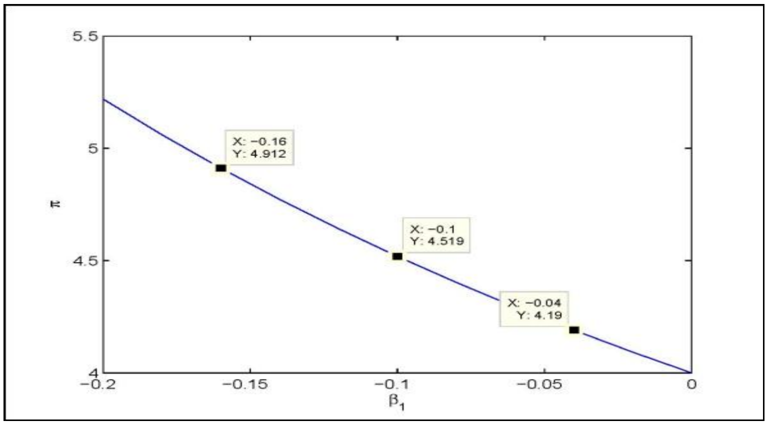


Figure 2

Relationship between interest rate r and the optimal proportion π^*



the volatility of the production σ. Here π decrease down to 0. That means the high volatility leads to more capital investment in the bank and less investment in the project. Taking $\sigma = 0.6$ for example, here π^*=1.494, i.e., the investor's optimal choice is to borrow from bank an amount about half of his wealth and invest all capitals in the corporate project. When $\sigma = 0.8$, π^*=0.5486, that is, the investor would only devote about half of his wealth to the corporate project and keep the rest in the bank.

Fixing $\sigma = 0.5$, π^* decreases with r. The curve is just as what we expected. π^* is a linear function with respect to r. That means higher rate in the bank can stimulate the risk-less investment in the bank.

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