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An Intertemporal Capital Asset Pricing Model under Incomplete Information

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ABSTRACT

In this paper, we use the classical dynamic programming principle to obtain the Hamilton- Jacobi-Bellman equation to derive the equilibrium market equation for all investors. Our paper derives the extended equilibrium market equation and the security market line of the classical capital asset pricing model of Merton (1987) in continuous time. It provides the continuous time analog to Merton's (1987) security market line. We derive the equilibrium market equation and the continuous time security market line of the intertemporal capital asset pricing model with incomplete information.

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I. INTRODUCTION

The capital asset pricing model of Sharpe-Lintner-Mossin, CAPM, is regarded as one of the most common developments in modern capital market theory. The CAPM model is still subject to theoretical and empirical criticism. In fact, since the model assumes the mean-variance criterion, it is subject to all the well-known theoretical objections to this criterion.

Merton (1973) develops an equilibrium model of the capital market. He shows that portfolio behavior for an intertemporal maximizer will be different when he faces a changing investment opportunity set instead of a constant one. Merton’s intertemporal model is based on consumer investor behavior and captures effects, which would never appear in a static model. These effects cause significant differences in specification of the equilibrium relationship among asset yields that appears in this model and the classical model.

By relaxing the main assumptions used in the CAPM, the model has been extended to more general economies. Merton’s (1973) model states that the expected excess return on any asset is given by a “multi-beta" version of the CAPM with the number of betas being equal to one plus the number of state variables.

In the same context, Breeden (1979) shows that Merton’s multi-beta pricing equation can collapse into a single beta equation where the expected excess return on any security, is proportional to its beta, with respect to aggregate consumption alone.

Since the acquisition of information and its dissemination are central activities in finance, and in capital markets, Merton (1987) develops a model of capital market equilibrium with incomplete information, CAPMI, to provide some insights into the behavior of security prices. He also studies the equilibrium structure of asset prices and its connection with empirical anomalies in financial markets.

Merton’s (1987) model is a two period model of capital market equilibrium in a costly economy where each investor has information about only a subset of the available securities. The key behavioral assumption is that an investor considers including security S in his portfolio only if he has some information on this security. Information costs have two components: the costs of gathering and processing data, and the costs of information transmission. This problem is related to the literature on the principal-agent problem, to the signaling models, to the differential information models and to the theory of generic and neglected stocks.

Merton’s model, the CAPMI, is an extension of the CAPM to a context of incomplete information. As Merton explains “Even modest recognition of institutional structures and information costs can go a long toward explaining financial behavior …”, the model also gives a general method for discounting future cash flows under uncertainty. Note that under complete information, the CAPMI model reduces to the standard CAPM.

Financial models based on complete information might be inadequate to capture the complexity of rationality in action. Some factors and constraints, like entry into the dealer business are not costless and may influence the short run behavior of security prices. Hence, most models developed in financial economics do not explicitly provide a functional role for the complicated and dynamic system of dealers, market makers and traders.

Besides, the treatment of information and its associated costs play a central role in capital markets. If an investor does not know about a trading opportunity, he will not act to implement an appropriate strategy to benefit from it. However, the investor must determine if potential gains are sufficient to warrant the costs of implementing the strategy.

From Merton's model (1987), it appears that taking into account the effect of incomplete information on the equilibrium price of an asset is similar to applying an additional discount rate to this asset's future cash flows. In fact, the expected return on the asset is given by the appropriate discount rate that must be applied to its future cash flows.

In that context, the investor's set is incomplete when it does not contain full information on expected rate of return and return variability. In the Merton's framework, no stock is held on which the investor does not have complete information. This has the potential to explain why individual and institutional investors do spend huge amounts of money in research and development activities, securities and information analysis before deciding to include an asset in their portfolios.

Merton (1987) adopts most of the assumptions of the original CAPM and relaxes the assumption of equal information across investors. Besides, he assumes that investors hold only securities of which they are aware. This assumption is motivated by the observation that portfolios held by actual investors include only a small fraction of all available traded securities.

In Merton's (1987) model, the expected returns increase with systematic risk, firm-specific risk, and relative market value. The expected returns decrease with relative size of the firm's investor base, referred to in Merton's model as the "degree of investor recognition".

The model shows that an increase in the size of the firm's investor base will lower investors' expected return and, all else equal, will increase the market value of the firm's shares. The main distinction between Merton's model and the standard CAPM is that investors invest only in the securities about which they are "aware". This assumption is referred to as incomplete information. However, the more general implication is that securities markets are segmented.

Merton's model is based on the assumption that there are several factors in addition to incomplete information that may explain this behavior for individuals and institutions. Hence, the presence of prudent-investing laws and traditions and other regulatory constraints can rule out investment in a particular firm by some investors. Using this assumption, Merton shows that the expected return depend on other factors in addition to market risk.

The main intuition behind this result is that the absence of a firm-specific risk component in the CAPM comes about because such risk can be eliminated (through diversification) and is not priced. It appears from Merton's model that the effect of incomplete information on expected returns is greater the higher the firm's specific risk and the higher the weight of the asset in the investor's portfolio. The effect of Merton's non-market risk factors on expected returns depend on whether the asset is widely held or not.

The intuition behind Merton's model is that investors consider only a part of the opportunity set, and that full diversification is not possible and that firm specific risk is priced in equilibrium.[1]

We describe in this paper the Capital market structure, asset value and the economic model. We use the classical dynamic programming principle to obtain the Hamilton-Jacobi-Bellman equation. This allows to derive the equilibrium market equation for all investors. Two cases are studied: the constant investment opportunity set and the general case.

Our analysis derives the extended equilibrium market equation and the security market line of the classical capital asset pricing model of Merton (1987) in continuous time. We provide the continuous time analog to Merton's (1987) security market line. The assumption of a constant investment opportunity set represents a sufficient condition for investors to behave as if they were single-period maximizers. It is also sufficient for the equilibrium return relationship specified by Merton's (1987) simple capital asset pricing model to obtain. We show that a generalization of Merton's (1973) 'multi-beta' asset pricing model obtains in this economy in the presence of shadow costs of incomplete information.

We derive the corresponding equilibrium market equation and the continuous time security market line of the intertemporal capital asset pricing model with incomplete information.

Our analysis provides two central results. The first result indicates that in the presence of a riskless asset and information costs regarding the n risky assets in the economy, (i) there exists a unique pair of efficient portfolio known as mutual funds, and (ii) the return distribution on the risky fund is log-normal.

The second result is a "Three Funds" theorem. It shows that all individuals in our economy within information uncertainty, regardless of their preferences, may attain their optimal portfolio positions by investing in at most 3 funds. These funds may be chosen to be: (i) the instantaneously riskless asset; (ii) the asset having the highest correlation with the state variable; and (iii) the market portfolio.

The structure of the paper is as follows. Section II presents our intertemporal model and the optimal portfolio under incomplete information. Section III provides some explicit optimal solutions for the case of CRRA utility functions.

II. THE ECONOMIC MODEL AND THE OPTIMAL PORTFOLIO UNDER INCOMPLETE INFORMATION

A. Capital Market Structure, Asset Value and the Economic Model

The model is based on the standard assumptions of a perfect market and continuous trading. Prices of assets follow Ito processes (continuous and not differentiable). Under the assumptions of continuous trading and a Markov structure, the first two moments of the distributions are sufficient statistics. It is assumed that there are n risky assets and one "instantaneously risk-less" asset. The riskless asset corresponds to the borrowing and lending rate on short government bonds.

The model in this paper is very similar in spirit to the models in Merton (1971, 1973), Breeden (1979), Adler and Dumas (1983) and Bellalah and Bellalah (2003). In the interest of brevity, common facets of this model will only be sketched. The unfamiliar reader may refer to those early developments of the model. Investors are price takers in perfectly competitive capital markets. They can trade continuously and trading takes place only at equilibrium prices. In terms of Merton (1973) terminology, the investment opportunity set may be stochastic. The state variables need not be restricted in number.

As usual, to be consistent with general equilibrium, prices must be recognized to be endogenously determined using supply and demand. All random shocks may affect both the supplies and demands for assets. All these shocks to the economy are captured as elements of the state vector. This vector describes the state of the world. For example asset prices and dividends can depend on time and the state variables.

We consider an economy in which there are K investors in the markets. Each investor can invest his wealth in n risky assets, (stocks). The prices of these assets satisfy the following dynamics

$$\begin{cases} dP_i(t) = P_i(t)[b_i + \lambda_i]dt + P_i(t)\sigma_i dB_i(t), \\ P_i(0) = P_i, \qquad i = 1,2,...,n \end{cases} \tag{1}$$

There is a riskless asset, (a bond) whose price satisfies the following dynamics:

$$dP_0(t) = rP_0(t)dt, \qquad P_i(0) = p_0 \tag{2}$$

where b_i represents the instantaneous expected rate of return for different stocks, λ_i is the information cost of asset i. The term σ_i is the instantaneous volatility and γ is the interest rate. They are all assumed to be bounded. The terms $B_1(t), B_2(t), \cdots, B_n(t)$ are n one-dimensional mutually dependent Brownian motions. They represent the external sources of uncertainty in the markets with correlation coefficients $\rho_{i,j}$, for $i, j = 1,2,\cdots n$.

At each moment, the k th investor, $k = 1, 2, \cdots, K$ can invest his money in the various assets. We denote by $W^k(t)$ his wealth and by x_i^k, $i = 1, 2, \cdots, n$ the proportion of his wealth in i th stock. The term $c^k(t)$ is the consumption rate, so the wealth of the k th investor satisfies the following accumulation equation:

$$\begin{cases} dW^k(t) = W^k(t)[\sum_{i=1}^{n} x_i^k(b_i + \lambda_i - r) + r]dt \\ \qquad\qquad - c^k(t)dt + W^k(t)\sum_{i=1}^{n} x_i^k \sigma_i dB_i(t) \\ W^k(0) = W_0^k \end{cases} \tag{3}$$

As in Breeden (1979), it is possible that fluctuations in some of the elements of the state vector do not affect any individual's expected utility of lifetime consumption, given the individual's wealth. A distinction can be made between state variables that affect at least one individual's expected utility, given his wealth. In this case, we define the state vector, s that contains those state variables that do affect at least one individual's expected utility given his wealth. Each individual's expected utility of lifetime consumption may be written as a function of his wealth, the vector of state variables, and time. The state variables are referred to as the state vector or as the vector of state variables.

We introduce one relevant state variable S, whose dynamic is

$$\begin{cases} dS(t) = S(t)bdt + S(t)\sigma db(t), \\ S(0) = s_0, \end{cases} \tag{4}$$

where $B(t)$ is one-dimensional Brownian motion which is dependent on $B_i(t)$ with correlation coefficients $\rho_{0,i}$, for $i = 1, 2, \cdots n$.

In this standard literature, each investor is assumed to maximize the expected value at each instant of a time additive and state-independent von Newmann-Morgenstern utility function for lifetime consumption. A quasi-concave utility and bequest functions of consumption and terminal wealth are used. At each instant, the individual k chooses an optimal rate of consumption and an optimal portfolio of risky assets.

The investor chooses his portfolio and consumption rate to maximize the following expected utility function

$$E[\int_0^T U^k(c^k(t), S(t), t)dt + h)(W^k(T), S(T)] \tag{5}$$

We denote by $J^k(W^k, S, t)$ the maximum expected utility of lifetime consumption that is obtainable with wealth and opportunities S at time t.

From the classical dynamic programming principle, we can obtain the Hamilton-Jacobi-Bellman equation.

$$\begin{cases} \frac{\partial J^k}{\partial t} + \max_{(x_i^k, c^k)} \{U^k(c^k, S, t) + \frac{\partial J^k}{\partial W^k} W^k [\sum_i^n x_i^k (b_i + \lambda_i - r) + r] \\ -\frac{\partial J^k}{\partial W^k} c^k + \frac{\partial J^k}{\partial S} bS + \frac{1}{2} \frac{\partial^2 J^k}{\partial (W^k)^2} (W^k)^2 \sum_{i=1}^n \sum_{j=1}^n x_i^k x_j^k \sigma_i \sigma_j \rho_{i,j} \\ + \frac{1}{2} \frac{\partial^2 J^k}{\partial S^2} S^2 \sigma^2 + \frac{\partial^2 J^k}{\partial W^k \partial S} W^k \sum_{i=2}^n x_i^k \sigma_i \sigma S \rho_{0,j} \} = 0, \quad t \in [0, T] \\ J^k(W^k, S, T) = h(W^k, S) \end{cases} \tag{6}$$

From $n+1$ first-order conditions, we have the equations for the optimal $c^k = c^k(W^k, S, t)$ and $x_i^k = x_i^k(W^k, S, t)$, $i = 1, 2, \cdots n$. In fact, first order conditions for an interior maximum may be stated as:

$$\begin{aligned} & U_c^k(c^k, S, t) = J_w^k(W^k, S, t) \\ & \frac{\partial J^k}{\partial W}(b_i + \lambda_i - r) + \frac{\partial^2 J^k}{\partial (W^k)^2} W^k \sum_{j=1}^n x_j^k \sigma_i \sigma_j \rho_{i,j} + \frac{\partial^2 J^k}{\partial W \partial S} S \sigma_i \sigma \rho_{0,i} = 0 \end{aligned} \tag{7}$$

The first equation is the usual intertemporal envelope condition to equate the marginal utility of current consumption to the marginal utility of future consumption (wealth). The second equation shows the linearity in the portfolio demands.

We denote the kth investor's demand function as $d_i^k = x_i^k W^k$, so we have

$$d_i^k = x_i^k W^k = A^k \sum_{j=1}^{n} v_{i,j}(b_j + \lambda_j - r) + H^k \sum_{j=1}^{n} \sigma_j \sigma S \rho_{0,j} v_{i,j}, \quad i = 1,2,\cdots,n \tag{8}$$

where the $v_{i,j}$ are the elements of the inverse matrix of the instantaneous variance-covariance matrix of return, $\Omega = (\sigma_{i,j})$, $\sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$, and

$$\begin{aligned} A^k &= -\frac{J_w^k}{\dfrac{\partial^2 J^k}{\partial W^2}} = -\frac{U_c}{U_{cc}\dfrac{\partial c^k}{\partial W^k}} \\ H^k &= -\frac{\dfrac{\partial^2 J^k}{\partial W \partial S}}{\dfrac{\partial^2 J^k}{\partial W^2}} = -\frac{\dfrac{\partial c^k}{\partial S}}{\dfrac{\partial c^k}{\partial W^k}} \end{aligned} \tag{9}$$

Equation (8) can be used to show that all investors' optimal portfolios can be represented as a combination of some portfolios or mutual funds. Condition (8) provides the individual's optimal risky asset portfolio in the presence of information uncertainty. The conditions provide the individual's optimal risky asset portfolio. They state that the indirect marginal utility of another unit of consumption must equal the indirect marginal utility of wealth for an optimal policy. Using these expressions for A^k and H^k, the demand function in equation (8) can be seen as having two components. The first term $A^k \sum_{j=1}^{n} v_{i,j}(b_j + \lambda_j - r)$ is the standard demand function for a risky asset by a single period mean-variance maximizer. The term A indicates the reciprocal of the investor's absolute risk aversion. The second term $H^k \sum_{j=1}^{n} \sigma_j \sigma S \rho_{0,j} v_{i,j}$ indicates the investor demand for an asset as a vehicle to hedge against "unfavorable" shifts in the investment opportunity set.

Some further results could be gained by restricting the class of utility functions. We can also add some simplifying assumptions to restrict the structure of the opportunity set.

In the following analysis, we derive the equilibrium market equation for all investors. We consider two cases: the constant investment opportunity set and the general case

First case: the case of a constant investment opportunity set

In this situation, the distribution of prices is lognormal for all assets. The assumption of a constant investment opportunity set is a sufficient condition for investors to behave as if they were single period maximizers. It is also a sufficient condition for the equilibrium return relationship specified by the CAPMI of Merton (1987) to obtain.

We assume that all risky assets are independent of preference, i.e. the state variable, at this case, $\rho_{0,j} = 0$, $j = 1, 2, \cdots n$. So equation (8) indicating the demand for the i th asset by the k th investor reduces to

$$d_i^k = x_i^k W^k = A^k \sum_{j=1}^{n} v_{i,j}(b_j + \lambda_j - r), \quad i = 1,2,\cdots,n \tag{10}$$

This demand corresponds also to the same demand that a one-period risk-averse mean-variance investor would have. In the presence of homogeneous expectations, about the opportunity set, the ratio of the demands for risky assets will be independent of preferences, and the same for all investors. Further, similar to the Theorem 1 in Merton, we have the following theorem:

Theorem 1: Consider an economy with n risky assets whose returns are log-normally distributed. In the presence of a riskless asset and information costs regarding the n risky assets, we have the following results: (i) there exists a unique pair of efficient portfolios known as mutual funds: the first one contains only the riskless asset and the second comprises only risky assets. These portfolios are independent of preferences, wealth distribution, or time horizon. All investors will be indifferent between choosing portfolios from among the original $(n+1)$ assets or from these two funds in the presence of incomplete information; (ii) the return distribution on the risky fund is log-normal; and (iii) the weight of the risky fund's assets invested in the k_{th} asset is given by the following expression:

$$\frac{\sum_{j=1}^{n} v_{k,j}(b_j + \lambda_j - r)}{\sum_{i=1}^{n}\sum_{j=1}^{n} v_{i,j}(b_j + \lambda_j - r)} \quad (k = 1,2,\cdots,n).$$

This theorem represents a continuous-time version of the separation theorem in Markowitz-Tobin and Merton (1987). The holdings in the risky portfolio indicate the optimal combination of risky assets. Then we let the aggregate demand functions $D_i = \sum_{k=1}^{k} d_i^k$, and $A = \sum_{k=1}^{k} A^k$, $x_i = \dfrac{D_i}{M}$, where M is the (equilibrium) value of all assets i.e. the market value, so

$$x_i M = D_i = A \sum_{j=1}^{n} v_{i,j}(b_j + \lambda_j - r), \quad i = 1,2,\cdots,n$$

and we have

$$b_i + \lambda_i - r = \frac{M}{A} \sum_{j=1}^{n} x_j \sigma_i \sigma_j \rho_{i,j}, \quad i = 1,2,\cdots,n \tag{11}$$

We define $b_M = \sum_{i=1}^{n} x_i (b_i - r)$ as the expected return rate on the market portfolio, $\lambda_m = \sum_{i=1}^{n} x_i \lambda_i$ as the information cost rate on the market and $\sigma_{i,M} = \sum_{i=1}^{n} x_j \sigma_i \sigma_j \rho_{i,j}$ as the covariance of the return on the i th stock with the return on the market portfolio, $\sigma_M^2 = \sum_{j=1}^{n} x_j \sigma_{j,M}$ as the variance of the market portfolio respectively. Then we have

$$b_i + \lambda_i - r = \frac{M}{A} \sigma_{i,M}, \quad i = 1,2,\cdots,n \tag{12}$$

Using the condition that the market portfolio is efficient in equilibrium, we can show that the equilibrium returns will satisfy "Merton's (1987)" simple model of capital market equilibrium with incomplete information.

Multiplying (12) by x_i and summing gives:

$$b_M + \lambda_M - r = \frac{M}{A} \sigma_M^2 \tag{13}$$

and

$$b_i + \lambda_i - r = \beta_i (b_M + \lambda_M - r) \tag{14}$$

where $\beta_i = \frac{\sigma_{i,M}}{\sigma_M^2}$ is the covariance of the return on the i th asset with the return on the market portfolio.

This is the extended equilibrium market equation and the security market line of the classical capital asset pricing model of Merton (1987) in continuous time. In fact, this is the continuous time analog to Merton's (1987) security market line. The assumption of a constant investment opportunity set represents a sufficient condition for investors to behave as if they were single-period maximizers. It is also sufficient for the equilibrium return relationship specified by Merton's (1987) simple capital asset pricing model to obtain.

Second case: The general case

Unfortunately, the assumption of a constant investment opportunity set is not consistent with the facts. In practice, there is at least one element of the opportunity set which is directly observable. This is the case for the interest rate. The effect of a changing interest rate is often considered as a single instrumental variable representing shifts in the opportunity set.

This is the general case. We assume that there exists an asset (by convention, the nth one) whose expected return shows the maximum correlations with the state variable

S. We use the same definitions as those in the previous case.

We also let $H = \sum_{k=1}^{k} H^k$, from (8), we have

$$x_i M = A \sum_{j=1}^{n} v_{i,j}(b_j + \lambda - r) + H \sum_{j=1}^{n} \sigma_j \sigma S \rho_{0,j} v_{i,j} \quad (15)$$

Substituting (15) into (8), we get

$$x_i^k W^k = \frac{A^k}{A} x_i M + (H^k - \frac{HA^k}{A}) \sum_{j=1}^{n} \sigma_j \sigma S \rho_{0,j} v_{i,j}, \quad i = 1,2,\cdots,n \quad (16)$$

This equation extends equation (A.2) in Breeden (1979) to account for the effects of information uncertainty. It provides the basis for the following allocation theorem which results immediately from individuals' portfolio demands. But, our theorem is a three fund theorem in the presence of information uncertainty.

Theorem 2. ("Three Funds" theorem): All individuals in our economy within information uncertainty, regardless of their preferences, may attain their optimal portfolio positions by investing in at most 3 funds. These funds may be chosen to be: (i) the instantaneously riskless asset; (ii) the asset having the highest correlation with the state variable; and (iii) the market portfolio.

Using equation (15), we can solve for the equilibrium expected returns on the individual assets. In this context, we obtain the following equation

$$b_i + \lambda_i - r = \frac{M}{A} \sum_{j=1}^{n} x_j \sigma_i \sigma_j \rho_{i,j} - \frac{H}{A} \sigma_i \sigma S \rho_{0,i}$$

We let $\sigma_{i,s} = \sigma_i \sigma S \rho_{0,i}$ as the covariance of the return on the ith stock with the state variable and $\sigma_{M,S} = \sum_{i=1}^{n} x_i \sigma_{i,S}$ as the covariance of the return on the market portfolio with the state variable. So the equilibrium expected returns on the individual assets can be written as:

$$b_i + \lambda_i - r = \frac{M}{A} \sigma_{i,M} - \frac{H}{A} \sigma_{i,s}, \quad i = 1,2,\cdots,n \quad 17)$$

It is possible to show that a generalization of Merton's (1973) 'multi-beta' asset pricing model obtains in this economy in the presence of shadow costs of incomplete information. In fact, the model obtains in this economy when betas are measured with respect to aggregate wealth and the returns of assets that hedge against changes in the various state variables, we do the following: aggregate individuals' portfolio demands and substitute in equilibrium expected excess returns for the market portfolio $b_M + \lambda_M - r$ and for assets perfectly correlated with the state variable $b_n + \lambda_n - r$.

Assuming that these assets exist and multiplying (17) by x_i and summing gives,

$$b_M + \lambda_M - r = \frac{M}{A}\sigma_M^2 - \frac{H}{A}\sigma_{M,S} \tag{18}$$

and also, we have

$$b_n + \lambda_n - r = \frac{M}{A}\sigma_{n,M} - \frac{H}{A}\sigma_{n,S} \tag{19}$$

So,

$$b_i + \lambda_i - r = (\sigma_{i,M}\ \sigma_{i,S})\begin{pmatrix} \sigma_M^2 & \sigma_{M,S} \\ \sigma_{n,M} & \sigma_{n,S} \end{pmatrix}^{-1}\begin{pmatrix} b_M + \lambda_M - r \\ b_n + \lambda_n - r \end{pmatrix} \tag{20}$$

This equation reveals that in equilibrium, investors are compensated in terms of expected returns, for bearing market risk (or systematic risk). They are compensated also for bearing the risk of unfavorable shifts in the investment opportunity set. This equation is a natural generalization of the results in the standard security market line and the results in Merton's (1987) CAPMI. We also can write equation (20) as

$$b_i + \lambda_i - r = \beta_{i,MSn}\begin{pmatrix} b_M + \lambda_M - r \\ b_n + \lambda_n - r \end{pmatrix} \tag{21}$$

where

$$\beta_{i,MSn} = (\sigma_{i,M} \quad \sigma_{i,S})\begin{pmatrix} \sigma_M^2 & \sigma_{M,S} \\ \sigma_{n,M} & \sigma_{n,S} \end{pmatrix}^{-1}$$

This is the corresponding equilibrium market equation and the continuous time security market line of the intertemporal capital asset pricing model with incomplete information. The term $\beta_{i,MSn}$ corresponds to the matrix of "multiple-regression" betas for all assets on the market and on the assets which are perfectly correlated with the state variables.

It is important to note that this "fundamental valuation equation" may be derived for any asset by using Ito's Lemma to find its expected instantaneous return from the asset price function, and then, by equating this drift rate to the equilibrium drift rates implied by the multi-beta model of (21). We can provide some explicit solutions for the case of CRRA utility functions.

III. EXPLICIT OPTIMAL SOLUTION FOR CRRA UTILITY FUNCTION

In this section, we consider an investor who only invests in one category of risky assets, (stocks) for which the price P_1 satisfies,

$$\begin{cases} dP_1(t) = P_1(t)[b_1 + \lambda_1]dt + P_1(t)\sigma_1 dB_1(t) \\ P_1(0) = P_1 \end{cases} \tag{22}$$

and in one riskless asset, (the bond), whose price satisfies:

$$dP_0(t) = rP_0(t)dt, \quad P_0(0) = P_0 \tag{23}$$

where b_1 represents the instantaneous expected rate of return in the stock, λ_1 is the information cost rate, σ_1 is the instantaneous volatility and r is the interest rate. They are all assumed to be bounded. The term $B_1(t)$, is one -dimensional Brownian motion. It represents the external sources of uncertainty in the market.

At each moment, the investor can invest his money in these two kinds of assets. We denote by $W(t)$ his wealth and by x the proportion of his wealth in the stock. The term $(1-x)$ is the proportion in the bond, $c(t)$ is the consumption rate, so the wealth of the investor satisfies the following equation:

$$\begin{cases} dW(t) = W(t)[x(b_1 + \lambda_1 - r) + r]dt \\ \qquad - c(t)dt + xW(t)\sigma_1 dB_1(t) \\ W(0) = W_0 \end{cases} \tag{24}$$

We introduce one relevant state variable S, whose dynamic is

$$\begin{cases} dS(t) = S(t)bdt + S(t)\sigma dB(t), \\ S(0) = s_0, \end{cases} \tag{25}$$

where B(t) is one dimensional Brownian motion which is dependent on $B_1(t)$ with correlation coefficients $\rho_{0,1}$.

The investor wants to choose his strategy x and consumption rate c(t) to maximize the following expected utility function

$$J(W_0) = \max_{x,c} E[\int_0^T \frac{e^{-rt}}{1-a} c^{1-a}(t) S^{1-a}(t)dt + e^{-rT} \frac{W^{1-a}(T)S^{1-a}(T)}{1-a}], \tag{26}$$

where r and a are constants, $r > 0$, $0 < a < 1$.

We refer to this utility function as the Constant Relative Risk Aversion (CRRA) case. We want to obtain the explicit optimal proportion x^*, consumption rate $c^*(t)$ and value function for this case. The admissible strategy $(x^*, c^*(t))$ is called an optimal strategy which attains the maximum of $J(W_0)$.

The idea to get the optimal solution comes from the technique to solve celebrated LQ (linear quadratic) problems in optimal control theory. This method is developed in Wu and Xu (1996).

From (24) and (25), we first have

$$\begin{cases} dW^{1-a}(t) = W^{1-a}(t)(1-a)[x(b_1+\lambda_1-r)+r]dt - W^{a}(t)(1-a)d(t)dt \\ \qquad -\frac{1}{2}W^{1-a}(t)a(1-a)x^2\sigma_1^2 dt + W^{1-a}(t)(1-a)x\sigma_1 dB_1(t) \\ W^{1-a}(0) = W_0^{1-a} \end{cases} \tag{27}$$

and

$$\begin{cases} dS^{1-a}(t) = S^{1-a}(t)(1-a)bdt - \frac{1}{2}S^{1-a}(t)a(1-a)\sigma^2 dt + S^{1-a}(t)(1-a)\sigma dB(t), \\ S^{1-a}(0) = s_0^{1-a} \end{cases} \tag{28}$$

So

$$\begin{cases} d[W^{1-a}(t)S^{1-a}(t)] = W^{1-a}(t)S^{1-a}(t)[(1-a)(x(b_1+\lambda_1-r)+r)+(1-a)b \\ \qquad -\frac{1}{2}a(1-a)x^2\sigma_1^2 - \frac{1}{2}a(1-a)\sigma^2 + \rho_{0,1}(1-a)^2 x\sigma\sigma_1]dt \\ \qquad - W^{-a}(t)S^{1-a}(t)(1-a)c(t)dt + W^{1-a}(t)S^{1-a}(t)(1-a)\sigma dB(t) \\ \qquad + W^{1-a}(t)S^{1-a}(t)(1-a)x\sigma_1 dB_1(t) \\ W^{1-a}(0)S^{1-a}(0) = W_0^{1-a}S_0^{1-a} \end{cases} \tag{29}$$

We let Q(t) be a nonnegative deterministic smooth function satisfying Q(T)=1 whose dynamics will be given latter. Applying Itô's formula to $\frac{e^{-rt}}{1-a}W^{1-a}(t)S^{1-a}(t)Q(t)$ from 0 to T and taking expectation on both sides, we have

$$\begin{aligned} &E\frac{e^{-rT}}{1-a}W^{1-a}(T)S^{1-a}(T)Q(T) = \frac{1}{1-a}W_0^{1-a}S_0^{1-a}Q(0) \\ &+E\int_0^T\{\frac{e^{-rT}}{1-a}W^{1-a}(t)S^{1-a}(t)Q(t)[-r+(1-a)(x(b_1+\lambda_1-r)+r) \\ &+(1-a)b-\frac{1}{2}a(1-a)x^2\sigma_1^2-\frac{1}{2}a(1-a)\sigma^2+\rho_{0,1}(1-a)^2x\sigma\sigma_1] \\ &-\frac{e^{-rt}}{1-a}W^{-a}(t)S^{1-a}(t)Q(t)(1-a)c(t)+\frac{e^{-rt}}{1-a}W^{1-a}(t)S^{1-a}(t)\dot{Q}(t)\}dt. \end{aligned}$$

So we can write

$$
\begin{aligned}
J(W_0) &= \frac{1}{1-a} W_0^{1-a} S_0^{1-a} Q(0) \\
&+ \max_{(x,c)} E\int_0^T \{\frac{e^{-rt}}{1-a} S^{1-a}(t)[c^{1-a}(t) - (1-a)W^{a}(t)Q(t)c(t) \\
&- aW^{1-a} tQ^{1-\frac{1}{a}}(t)] + \frac{e^{-rt}}{1-a} W^{1-a}(t) S^{1-a}(t) Q(t)[-\frac{1}{2} a(1-a)\sigma_1^2 x^2 \\
&+ (\rho_{0,1}(1-a)^2 \sigma\sigma_1 + (1-a)(b_1 + \lambda_1 - r))x \\
&- \frac{(1-a)(\rho_{0,1}(1-a)\sigma\sigma_1 + b_1 + \lambda_1 - r)^2}{2a\sigma_1^2}] \\
&+ \frac{e^{-rt}}{1-a} W^{1-a}(t) S^{1-a}(t)[\dot{Q}(t) - arQ(t) + (1-a)bQ(t) \\
&- \frac{1}{2} a(1-a)\sigma^2 Q(t) + \frac{(1-a)(\rho_{0,1}(1-a)\sigma\sigma_1 + b_1 + \lambda_1 - r)^2}{2a\sigma_1^2} Q(t) + aQ^{1-\frac{1}{a}}(t)]\}dt \\
&= \frac{1}{1-a} W_0^{1-a} S_0^{1-a} Q(0) + I + II + III
\end{aligned}
$$

Here

$$
I = \max_{(x,c)} E\int_0^T \frac{e^{-rt}}{1-a} S^{1-a}(t)[c^{1-a}(t) - (1-a)W^{-a}(t)Q(t)c(t) - aW^{1-a}(t)Q^{1-\frac{1}{a}}(t)]dt
$$

$$
II = \max_{(x,c)} E\int_0^T \frac{e^{-rt}}{1-a} W^{1-a}(t) S^{1-a}(t) Q(t) L(x) dt
$$

where

$$
\begin{aligned}
L(x) = &-\frac{1}{2} a(1-a)\sigma_1^2 x^2 + (\rho_{0,1}(1-a)^2 \sigma\sigma_1 + (1-a)(b_1 + \lambda_1 - r))x \\
&- \frac{(1-a)(\rho_{0,1}(1-a)\sigma\sigma_1 + b_1 + \lambda_1 - r)^2}{2a\sigma_1^2}
\end{aligned}
$$

and

$$
\begin{aligned}
III = &\max_{(x,c)} E\int_0^T \frac{e^{-rt}}{1-a} W^{1-a}(t) S^{1-a}(t)[\dot{Q}(t) - arQ(t) + (1-a)bQ(t) \\
&- \frac{1}{2} a(1-a)\sigma^2 Q(t) + \frac{(1-a)(\rho_{0,1}(1-a)\sigma\sigma_1 + b_1 + \lambda_1 - r)^2}{2a\sigma_1^2} Q(t) + aQ^{1-\frac{1}{a}}(t)]dt
\end{aligned}
$$

Now we let $Q(t)$ be the solution of the following ordinary differential equation of Bernoulli type:

$$\begin{cases} -\dot{Q}(t) = MQ(t) + aQ^{1-\frac{1}{a}}(t) \\ Q(T) = 1, \quad t \in [0,T] \end{cases} \tag{30}$$

and

$$M = -ar + (1-a)b - \frac{1}{2}a(1-a)\sigma^2 + \frac{(1-a)(\rho_{0,1}(1-a)\sigma\sigma_1 + b_1 + \lambda_1 - r)^2}{2a\sigma_1^2} \tag{31}$$

Let $\tilde{Q}(t) = e^{-M(T-t)}Q(t)$, then

$$\begin{cases} -\dot{\tilde{Q}}(t) = ae^{-\frac{1}{a}M(T-t)}\tilde{Q}^{1-\frac{1}{a}}(t) \\ \tilde{Q}(T) = 1, \quad t \in [0,T] \end{cases}$$

So

$$\tilde{Q}(t) = [1 + \int_t^T e^{-\frac{1}{a}M(T-s)}ds]^a$$

and we obtain:

$$\begin{cases} Q(t) = e^{M(T-t)}[1 + \int_t^T e^{-\frac{1}{a}M(T-s)}ds]^a \quad t \in [0,T], \\ Q(0) = e^{MT}[1 + \int_0^T e^{-\frac{1}{a}M(T-s)}ds]^a \end{cases} \tag{32}$$

Then we can look back I and II. If we take

$$c^*(t) = (1-a)^{-\frac{1}{a}}Q^{-\frac{1}{a}}(t)W(t), \quad t \in [0,T] \tag{33}$$

where Q(T) is given by (32) and is positive. One can check that I attains its maximum at point $c^*(t)$ and I=0. This is also the feed back form of wealth.
Then we take:

$$x^* = \frac{\rho_{0,1}(1-a)\sigma\sigma_1 + b_1 + \lambda_1 - r}{a\sigma_1^2} \tag{34}$$

where the denominator is positive. One can check that $L'(x^*) = 0$ and $L''(x^*) < 0$. Thus the function L(x) attains its maximum at point x^* and $L(x^*)$=0, II=0.

So for CRRA case, we can have the explicit optimal proportion x^* from (34), the optimal consumption rate $c^*(t)$ from (33) and the optimal value function:

$$J(W_0) = \frac{1}{1-a} W_0^{1-a} S_0^{1-a} Q(0) \quad (35)$$

where $Q(0)$ is given by (32).

IV. SUMMARY

Information costs, which are different from transaction costs, are justified by the huge amounts of money spent by individual and institutional investors in analysis, valuing, and treating information. Information is fundamental for asset pricing since investor's information set is incomplete because it does not contain information regarding the expected return and its variability. These information costs are assimilated by Merton and here as additional discount rates for future cash flows. When information costs are ignored, our model reduces to the standard model.

An intertemporal model of the capital market has been developed which is consistent with both the expected utility maximization and the limited liability of assets. The analysis in Merton (1973) shows that the equilibrium relationships among expected returns specified by the classical capital asset pricing model will obtain only under some special additional assumptions. Merton's model is robust in the sense that it can be extended in an obvious way to include some other effects.

In this context, we derive an intertemporal capital asset pricing model in an economic context permitting both stochastic consumption-goods prices and stochastic portfolio opportunities. The model is a generalization of Merton's (1973) continuous-time model, deriving equivalent pricing equations that are simpler in form. Breeden (1979) concludes his paper by stressing the fact that areas that need additional theoretical development include the role of firms and their optimal investment and capital structure decisions, the impact of information costs and transaction costs.

We provide two theorems. Theorem 1 shows that in the presence of a riskless asset and information costs regarding the n risky assets in the economy,
(i) there exists a unique pair of efficient portfolio known as mutual funds: the first one contains only the riskless asset and the second comprises only risky assets. All investors will be indifferent between choosing portfolios from among the original $(n+1)$ assets or from these two funds in the presence of incomplete information; and (ii) the return distribution on the risky fund is log-normal.

Theorem 2 is a "Three Funds" theorem. It shows that all individuals in our economy within information uncertainty, regardless of their preferences, may attain their optimal portfolio positions by investing in at most 3 funds. These funds may be chosen to be: (i) the instantaneously riskless asset; (ii) the asset having the highest correlation with the state variable; and (iii) the market portfolio.

Our model is a generalization of Merton (1973) and Breeden (1976) by accounting for the effects of information costs.

ENDNOTES

1. Merton's model may be stated as follows: $\bar{R}_p - r = \beta_p[\bar{R}_m - r] + \lambda_p - \beta_p \lambda_m$, where

$\bar{R}_p$: the equilibrium expected return on security P; $\bar{R}_m$: the equilibrium expected return on the market portfolio; r: the riskless rate of interest; $\beta_p = \frac{cov(\tilde{R}_p / \tilde{R}_m)}{var(\tilde{R}_m)}$: the beta of security P, that is the covariance of the return on that security with the return on the market portfolio, divided by the variance of market return; λ_p : the equilibrium aggregate " shadow cost" for the security P. It is of the same dimension as the expected rate of return on this security P; λ_m : the weighted average shadow cost of incomplete information over all securities.

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