

## **Optimal Portfolios with Guarantee at Maturity: Computation and Comparison**

Jean-Luc Prigent<sup>a</sup> and Fabrice Tahar<sup>b\*</sup>

<sup>a</sup> *THEMA, University of Cergy,  
33 bd du Port, 95011, Cergy, France  
Jean-luc.Prigent@eco.u-cergy.fr*

<sup>b</sup> *THEMA, University of Cergy,  
33 bd du Port, 95011, Cergy, France  
fabricetahar@hotmail.com*

### **ABSTRACT**

Portfolio insurance allows investors to recover, at maturity, a given percentage of their initial capital. This limits downside risk in falling markets. Besides, it allows some participation in rising markets. One of the standard portfolio insurance methods is the *Constant Proportion Portfolio Insurance* (CPPI). We analyse options on cushion associated to CPPI. This kind of Power options corresponds in particular to the solution of a portfolio optimization problem in which an additional guarantee constraint must be satisfied at maturity. We also compare this strategy with the standard OBPI method

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## I. INTRODUCTION

Portfolio insurance is designed to give the investor the ability to limit downside risk while allowing some participation in upside markets. Such methods allow investors to recover, at maturity, a given percentage of their initial capital, in particular in falling markets. There exist various portfolio insurance models, among them the *Option Based Portfolio Insurance* (OBPI) and the *Constant Proportion Portfolio Insurance* (CPPI).

The OBPI, introduced by Leland and Rubinstein (1976), consists in a portfolio invested in a risky asset  $S$  (usually a financial index such as the S&P) covered by a listed put written on it. Whatever the value of  $S$  at the terminal date  $T$ , the portfolio value will be always greater than the strike  $K$  of the put. At first glance, the goal of the OBPI method is to guarantee a fixed amount only at the terminal date. In fact, as recalled and analyzed in this paper, the OBPI method allows to get a portfolio insurance at any time. Nevertheless, the European put with suitable strike and maturity may be not available on the market. Hence it must be synthesized by a dynamic replicating portfolio invested in a riskfree asset (for instance, T-bills) and in the risky asset.

The CPPI was introduced by Perold (1986) (see also Perold and Sharpe (1988)) for fixed-income instruments and Black and Jones (1987) for equity instruments. This method uses a simplified strategy to allocate assets dynamically over time. The investor starts by setting a floor equal to the lowest acceptable value of the portfolio. Then, he computes the cushion as the excess of the portfolio value over the floor and determines the amount allocated to the risky asset by multiplying the cushion by a predetermined multiple. Both the floor and the multiple are functions of the investor's risk tolerance and are exogenous to the model. The total amount allocated to the risky asset is known as the exposure. The remaining funds are invested in the reserve asset, usually T-bills.

The higher the multiple, the more the investor will participate in a sustained increase in stock prices. Nevertheless, the higher the multiple, the faster the portfolio will approach the floor when there is a sustained decrease in stock prices. As the cushion approaches zero, exposure approaches zero too. In continuous time, this keeps portfolio value from falling below the floor. Portfolio value will fall below the floor only when there is a very sharp drop in the market before the investor has a chance to trade.

The present paper analyses the properties of options defined on the cushion. In a first section, a generalized OBPI method is introduced, based on utility risk aversion. In a second section, we analyse the dynamic properties of such insured portfolio.<sup>1</sup> Finally, we provide simulations to better illustrate the properties of this portfolio.

## II. GENERALIZED OBPI

One of the main problems that we encounter in financial portfolio optimisation is the following: Which strategy is optimal for given investor's risk aversion and guarantee level?

Many previous studies on portfolio optimisation generally consider an investor who wants to maximise the expected utility of his wealth at maturity (see for example Cox and Huang (1989)). El Karoui, Jeanblanc and Lacoste (2005) show that, under

market completeness and for various utility functions, the payoff of the optimal portfolio looks like an option on a power of the underlying asset<sup>2</sup>. Prigent (2004) (see also Bertrand, Lesne and Prigent (2001)) provides a generalization of this result by introducing more general guarantee constraints.

Recall the main results. The financial market is assumed to be arbitrage free, complete, and without friction. Asset prices are supposed to follow diffusion processes. We assume also that there exists a riskless asset, denoted by  $B$  with an instantaneous rate  $r$ .

The portfolio value  $(V_t)_t$  is self-financing and the process  $(V_t e^{-r(T-t)})_t$  is a  $Q$ -martingale, where  $Q$  is the risk-neutral probability and the information  $F_t$  at time  $t$  is generated by the observation of asset prices up to time  $t$ .

Denote by  $\eta_t = E\left[\frac{dQ}{dP} \mid F_t\right]$  the Radon-Nikodym derivative of the probability  $Q$

with respect to the historical probability  $P$ . By absence of arbitrage, the initial budget constraint is:

$$V_0 = E_Q[V_T e^{-rT}] = E_P[V_T M_T] \quad (1)$$

where  $M_T = \eta_T e^{-rT}$ .

The investor maximises his expected utility defined on his terminal wealth, under the historical probability. As usual, the utility function  $U$  is supposed to be increasing and twice continuously differentiable.

#### A. Case Without Guarantee Constraint

Following Carr and Madan (2001), we have to search the optimal payoff  $h^*(S_T)$  of the portfolio at maturity. Therefore, the portfolio optimisation problem is:

$$\text{Max} E_P[h(S_T)] \text{ under } V_0 = e^{-rT} E_Q[h(S_T)] \quad (2)$$

This problem can be solved by using the Lagrange method. Denote by  $L$  the Lagrangian:

$$L = \int_0^{\infty} U[h(s)] f_P ds - \lambda \left[ \int_0^{\infty} h(s) e^{-rT} f_Q ds - V_0 \right] \quad (3)$$

where  $f_P$  and  $f_Q$  denote respectively the densities of the distribution of  $S_T$  with respect to the probabilities  $P$  and  $Q$ .

The first order condition, which is necessary and sufficient, is:

$$U'(h^*) = \lambda g \quad (4)$$

where  $\lambda$  is the Lagrange multiplier and  $g$  is the density of the Radon-Nikodym derivative of the risk-neutral probability with respect to the historical one.

Assuming that  $U'$  is invertible and denoting by  $J$  its inverse, we deduce:

$$h^* = J(\lambda g) \quad (5)$$

Thus, the optimal payoff depends on the risk aversion (through  $J$ ) and also on the difference between  $P$  and  $Q$  (through  $g$ ).

*Examine the following basic example:* the utility function is assumed to be of CRRA type:

$$U(x) = \frac{x^\alpha}{\alpha}, \quad 0 < \alpha < 1 \quad (6)$$

$$J(x) = (U')^{-1}(x) = x^{\frac{1}{\alpha-1}}$$

The stock price is assumed to follow a geometric Brownian motion. However, the problem is static: the investor trades only at initial time  $t=0$ .

Under the previous assumptions, it can be deduced that:

$$h^*(s) = \frac{V_0 e^{rT}}{\int_0^\infty \frac{g(s)^{\alpha-1} f_p(s) ds}{\alpha}} g(s)^{\frac{1}{\alpha-1}} \quad (7)$$

where  $f_p$  is the density of the Lognormal distribution of the stock price  $S_T$ .

Recall that for the Black and Scholes model,  $g(s)$  is given by:

$$g(s) = \psi s^{-k} \quad (8)$$

with:

$$\theta = \frac{\mu - r}{\sigma}, \quad A = -\frac{1}{2}\theta^2 T + \frac{\theta}{\sigma}(\mu - \frac{1}{2}\sigma^2)T, \quad k = \frac{\theta}{\sigma} \text{ and } \psi = e^A S_0^k \quad (9)$$

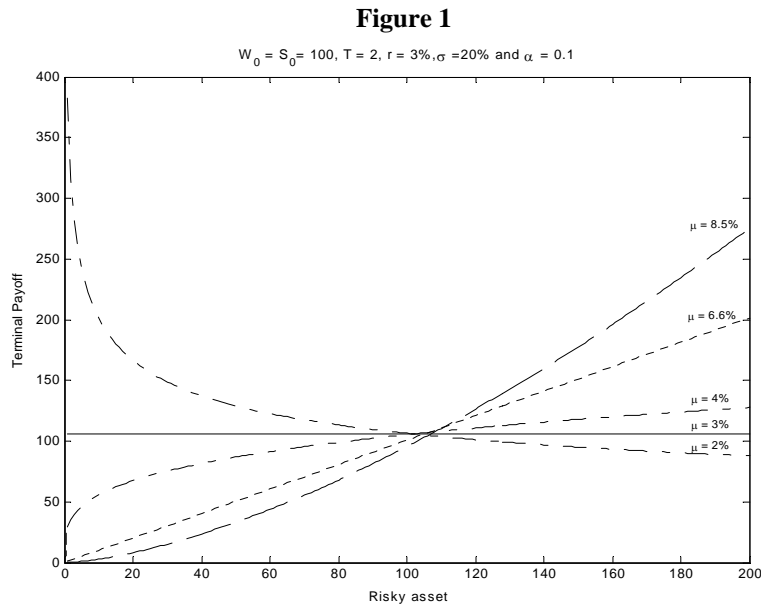
Therefore,  $h^*(s)$  is given by:

$$h^*(s) = ds^m \quad (10)$$

with

$$d = \frac{V_0 e^{rT}}{\int_0^{\infty} \frac{\alpha}{\alpha-1} f_P(s) ds} \psi^{\frac{1}{\alpha-1}} > 0 \quad \text{and} \quad m = \frac{k}{1-\alpha} \quad (11)$$

Thus,  $h^*(s)$  is an increasing function of the value  $s$  of the stock price at maturity. It is from depends only on the inverse of the relative risk aversion  $1-\alpha$  and the parameter  $k$  that looks like a Sharpe ratio. Figure 1 shows how the optimal payoff  $h^*$  depends on the instantaneous return  $\mu$  of the stock price.



### B. Case with Guarantee Constraint

Following Bertrand, Lesne and Prigent (2001) and Prigent (2004), we have to search the optimal payoff  $h^{**}(S_T)$  of the portfolio at maturity such that  $h^{**}(S_T) > h_c(S_T)$  where  $h_c(S_T)$  is a given constraint guarantee at maturity. For example, if  $h_c(S_T)$  is linear:  $h_c(S_T) = a S_T + b$ ,

Therefore, the portfolio optimisation problem is:

$$\text{Max} E_P[U(h(S_T))] \quad \text{under} \quad V_0 = e^{-rT} E_Q[h(S_T)] \quad \text{and} \quad h^{**}(S_T) > h_c(S_T) \quad (12)$$

This problem can be solved by using a generalization of Kuhn-Tucker theorem to infinite dimensional space. Then, we obtain (see Prigent (2004) for example):

$$h^{**}(S_T) = \text{Max}(h^*(S_T), h_c(S_T)) \quad (13)$$

where  $h^*(S_T)$  is an optimal solution for a non constrained portfolio with an initial value smaller than  $V_0$ .

Thus, the optimal payoff depends now on the risk aversion (through  $J$ ), on the difference between  $P$  and  $Q$  (through  $g$ ) and also from the guarantee constraint  $h_c$ .

**Remark 1:** The optimal solution  $h^{**}$  can be written also as:

$$h^{**} = h_c + \text{Max}(h^* - h_c, 0) \quad (14)$$

Therefore,  $h^{**}$  can be considered as a combination of a European Call with strike  $h_c$  and of the guarantee  $h_c$  itself.

### III. DYNAMIC PROPERTIES

Consider the case of Geometric Brownian motion (see previous basic example). As we have seen, the optimal solution without constraint has the following form:

$$h^*(S_T) = dS_T^m \quad (15)$$

In that case, the Call component  $\text{Max}(h^* - h_c, 0)$  looks like:

$$\text{Max}(h^*(S_T) - h_c(S_T), 0) = \text{Max}(dS_T^m - h_c(S_T), 0) \quad (16)$$

If the guarantee constraint  $h_c$  is constant then we get a Power option: here, a Call on the power,  $S_T^m$ .

Otherwise, we get polynomial options if for example  $h_c$  is linear; Note that we can also consider convex constraint such as  $h_c(S_T) = a\text{Max}(S_T - c, 0) + b$  if for example we search for a fixed guarantee  $b$  and a fixed percentage  $a$  of the underlying  $S_T$  if this one is higher than the amount  $c$ .

In order to compare the portfolio  $h^{**}$  with payoffs of standard portfolio insurance methods, we suppose in what follows that the constraint  $h_c$  is a constant denoted by  $K$ .

Recall that for the standard OBPI method, the payoff is given by:

$$K + \text{Max}(S_T - K, 0) \quad (17)$$

and, for the standard CPPI method, the payoff is given by  $K + d'S_T^m$ .

Finally, the payoff  $h^{**}$  is given here by:

$$K + \text{Max}(dS_T^m - K, 0) \quad (18)$$

Therefore, these three portfolios offer the same guarantee at maturity. For a given initial investment  $V_0$ ,  $K$  is determined from the relation

$$Ke^{-rT} + \text{Call}(K, T) = V_0 \quad (19)$$

where  $\text{Call}(K, T)$  is the value at time 0 of the European Call  $\text{Max}(S_T - K, 0)$  (for instance, evaluated by the Black and Scholes's formula).

Then, the parameters  $d$ ,  $m$ ,  $d'$  and  $m'$  are such that the initial values of both the CPPI payoff and the payoff  $h^{**}$  have the same price  $V_0$  as the OBPI payoff. Note that, if we impose the equality of the return expectations, then all these parameters are entirely determined<sup>4</sup>.

Notations:

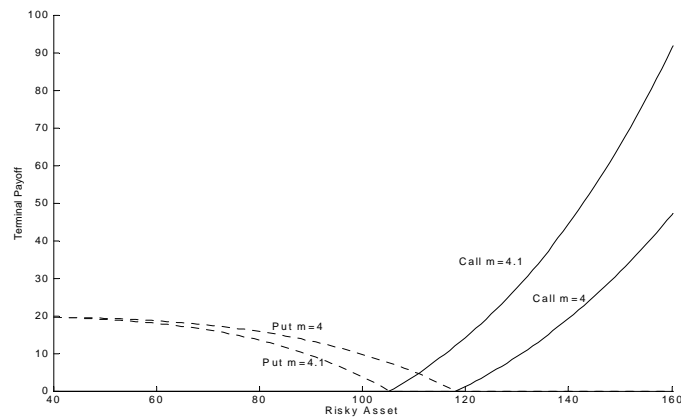
$$S'_0 = dS_0^m e^{\frac{1}{2}m(m-1)\sigma^2 T} \quad \text{and}$$

$$d'_1 = \frac{\text{Log}\left[\frac{S'_0}{K}\right] + \left[mr + \frac{1}{2}m^2\sigma^2\right]T}{m\sigma\sqrt{T}}, \quad d'_2 = d'_1 - m\sigma\sqrt{T} \quad (20)$$

where  $N(\cdot)$  is the cumulative distribution function of the Normalized Gaussian distribution and  $N'(\cdot)$  is its density function.

We illustrate in Figure 2 the optimal portfolio payoff  $h^{**}$ .

**Figure 2**



We examine now the “greeks” of the insured portfolio  $h^{**}$ .

First we determine the **Delta**:

$$\Delta = \frac{\partial V_0}{\partial S_0} = \frac{m}{S_0} N(d'_0) S_0 e^{(m-1)rT} \quad (21)$$

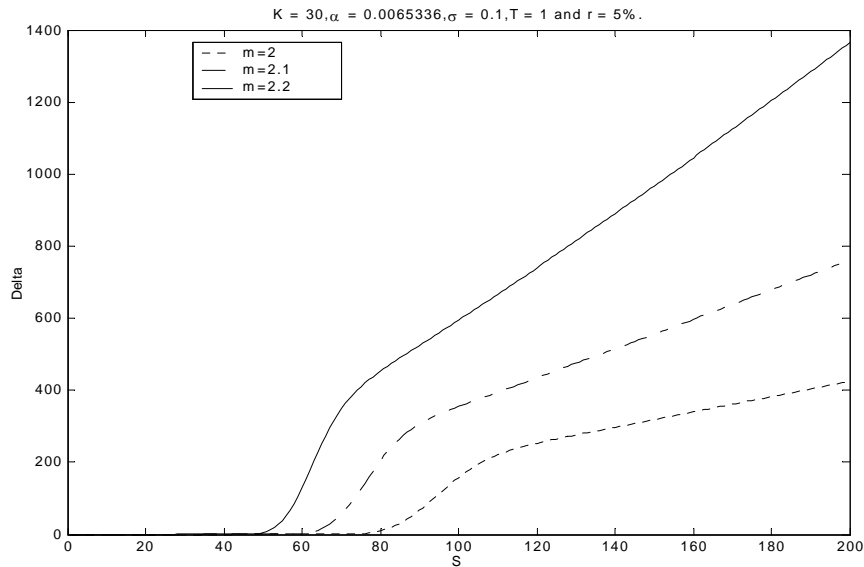
The Delta is an increasing function of the underlying asset  $S_T$ . The higher the power  $m$ , the more convex its curve. There exists also an inflection point from what the curve is almost linear.

We determine the Delta’s derivative (which is usually a measure of the impact of transaction costs). It is the **Gamma** with value equal to

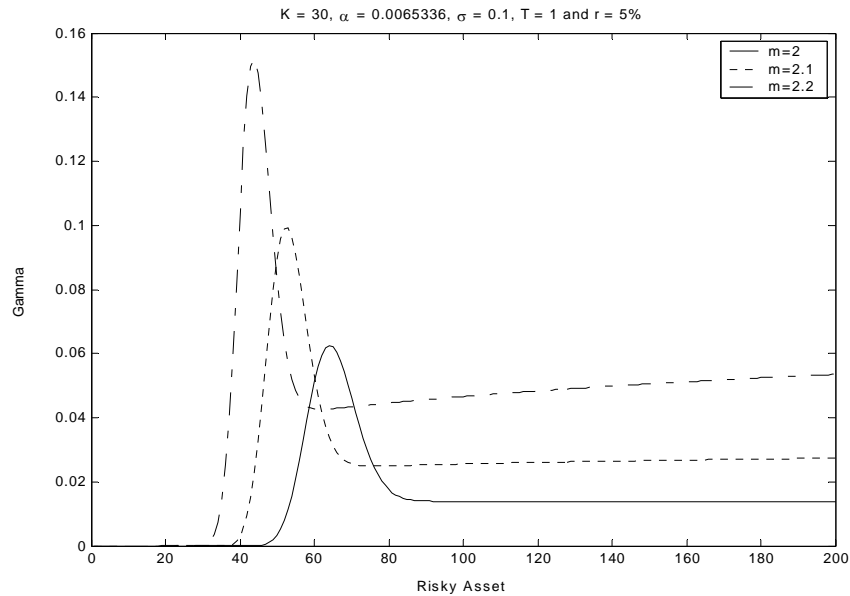
$$\Gamma = \frac{\partial^2 V_0}{\partial S_0^2} = \alpha m S_0^{m-2} e^{(m-1)(r + \frac{1}{2}m\sigma^2)T} \Gamma \left[ \frac{N'(d'_1)}{\sigma\sqrt{T}} + (m-1)N(d'_1) \right] \quad (22)$$

The Gamma is increasing for small values of  $S_T$ . Then it decreases. For high values of  $S_T$ , it looks like an affine curve the slope of which is an increasing function of the power  $m$ .

**Figure 3**





**Figure 4**

We analyze now the influence of the volatility: the **Vega**. We get the following formula:

$$\text{Vega} = \frac{\partial V_0}{\partial \sigma} = \alpha S_0^m e^{\frac{1}{2}\sigma^2(m-1)mT} \left[ \sqrt{T} m e^{(m-1)rT} N'(d_1') \right] \quad (23)$$

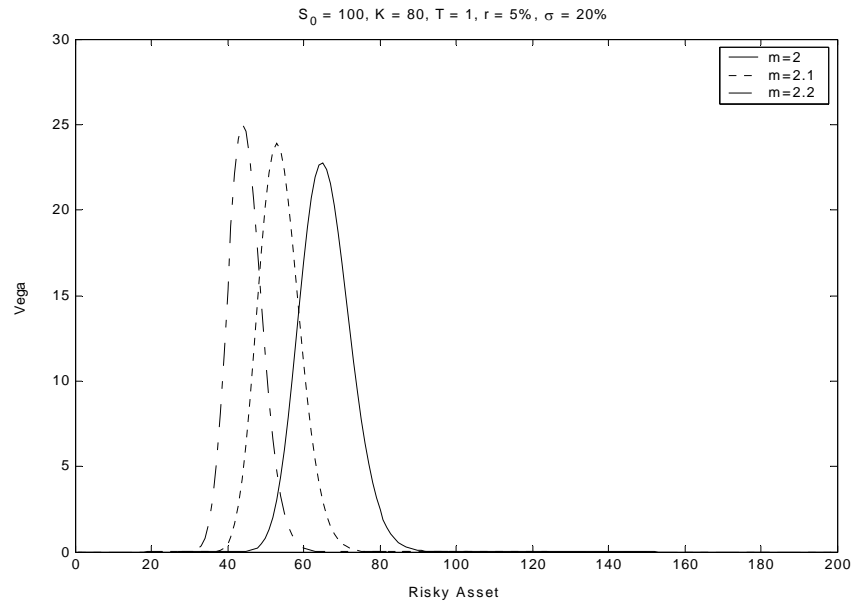
The vega is always positive. It is also increasing for small values of  $S_T$  and then it decreases. We can see also that its maximum value is increasing with the power  $m$ . Finally, we examine the influence of the power  $m$  itself.

#### Sensitivity to the power $m$ :

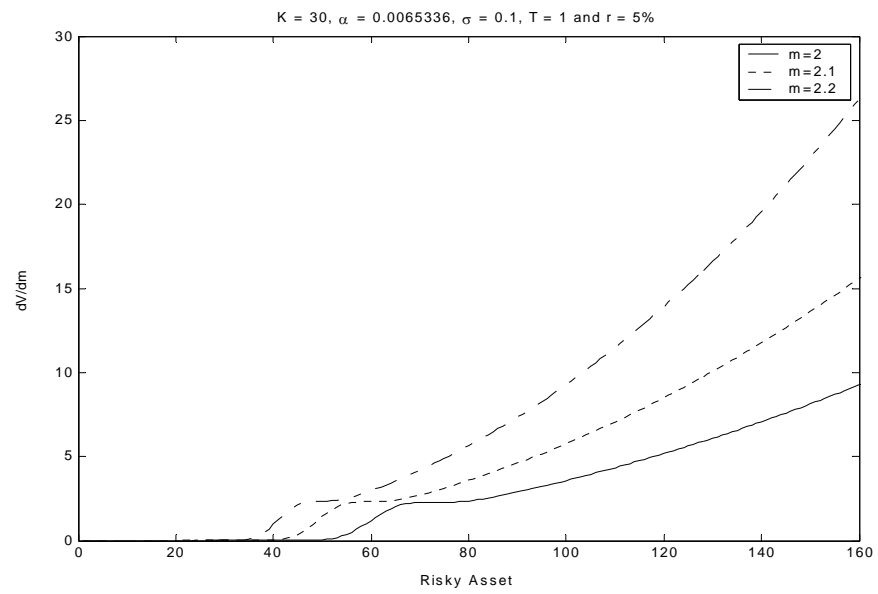
$$\frac{\partial V_0}{\partial m} = T e^{(m-1)rT} \left[ rBS + rK e^{-mrT} N(d_2') + \frac{\sigma S_0' N'(d_1')}{\sqrt{T}} \right] \quad (24)$$

where BS is the Black-Scholes price of the Call  $C(S_0', K, mr, m\sigma)$ . For fixed parameter  $d$ , the portfolio value is increasing with respect to the power  $m$ . Then it decreases. For high values of  $S_T$ , its slope is increasing with  $m$ .

**Figure 5**

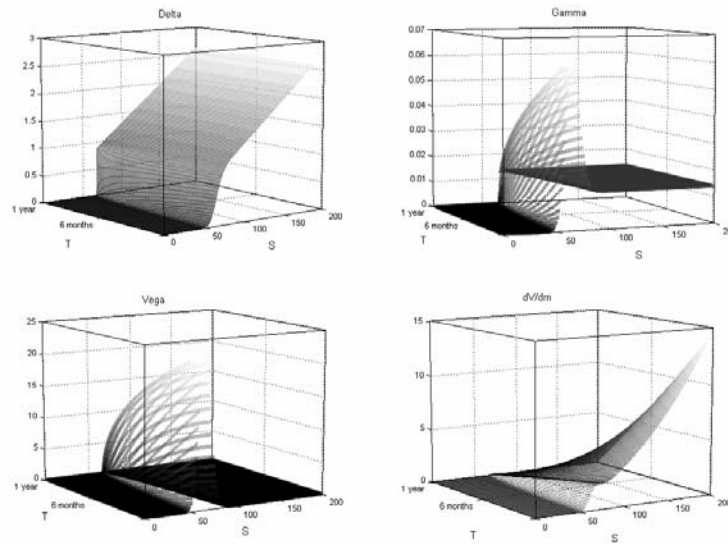


**Figure 6**



The following figure illustrates these sensitivities in a 3-dimensional space when both maturity and underlying values  $S_T$  vary.

**Figure 7**



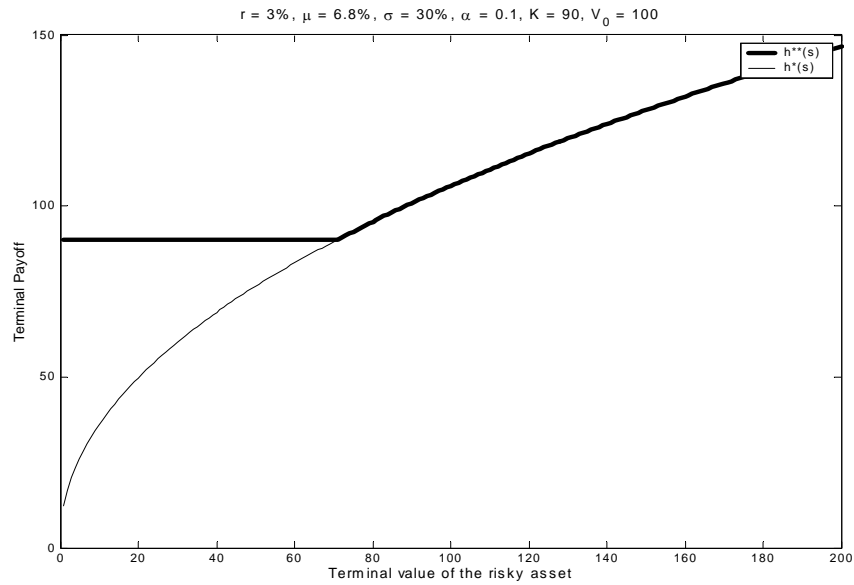
#### IV. SIMULATIONS

We examine here the optimal portfolio payoff  $h^{**}$  of an investor with a given risk aversion and a fixed guarantee at maturity. For example, if he has a high-risk aversion and if he wants to avoid a 10% loss in his investment then the optimal payoff  $h^{**}$  is concave as shown in Figure 8. Figure 9 provides examples of trajectories when using Power Call options and standard OBPI method. Three trajectories of  $S$  and three values of  $m$  are considered.

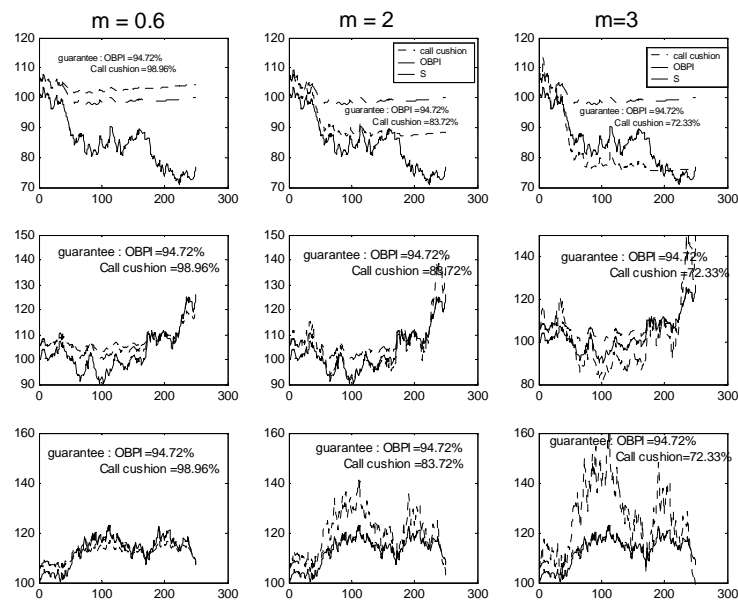
We note that the guarantee level depends much on the value of  $m$ . This result is rather intuitive since, in order to get a payoff with high convexity, the investor must accept to support higher losses when the underlying value  $S_T$  is low. Recall that the OBPI payoff corresponds to  $m=1$  ("linear" case).

The first column indicates the trajectory of the Power Call option when the optimal payoff  $h^{**}$  is concave ( $m=0.6 < 1$ ). When the financial market is bearish, the portfolio value reaches quickly the floor but remains above the OBPI payoff. When the market is bullish, the convexity of the payoff implies that the portfolio value is smaller than the OBPI's one.

**Figure 8**



**Figure 9**



The second column examines an intermediate case with a convex payoff  $h^{**}$ . Contrary to the previous case, the underlying variations are growing with underlying variations. The guarantee level is also smaller.

The third column exhibits the smallest guarantee and trajectories with high volatilities.

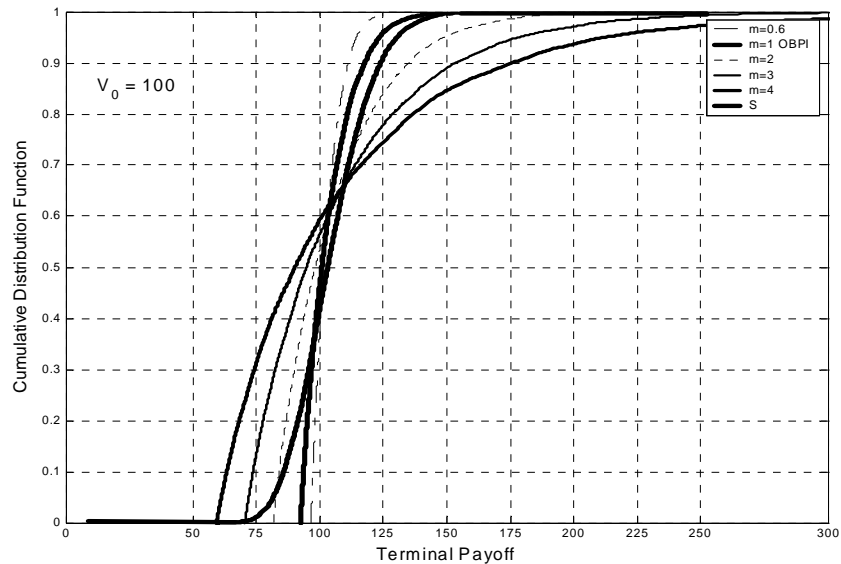
Table 1 provides the first fourth expectations and the guarantee levels. As it can be seen, the four expectations are increasing with respect to  $m$  contrary to the guarantee level.

**Table 1**

$V_0=100$	Guarantee level	$E[V_T]$	$\sigma[V_T]$	Skewness	Kurtosis
$m=0.6$ (concave)	98.96%	103.14	5.73	1.33	5.05
$m=1$ (linear :OBPI)	94.72%	103.71	9.93	1.43	5.60
$m=2$ (convex)	83.72%	104.88	22.06	1.60	6.17
$m=3$	72.33%	106.31	36.52	1.98	8.82
$m=4$	60.86%	108.06	55.07	2.61	14.02

Figure 10 illustrates the cumulative and density distribution functions of Power Call options.

**Figure 10**



All the cumulative distributions functions intersect which prove that no strategy dominates at the first order another one. We still observe that higher the potential gain, smaller the guarantee.

## V. CONCLUSION

We have analysed special kind of option corresponding to the solution of a portfolio optimization problem in which an additional guarantee constraint must be satisfied at maturity. We have compared this strategy with the standard OBPI method and illustrated more precisely the influence of risk aversion on the insured portfolio. Further extensions may introduce for example a stochastic volatility as in Bertrand and Prigent (2003).

## ENDNOTES

1. See Tahar (2005) for more details about proofs of formula.
2. See also Jensen and Sorensen (2001).
3. The guarantee constraint  $h_c$  may be stochastic: for example if  $h_c(S_T) = aS_T + b$ .
4. As shown in Prigent (2001) and Bertrand and Prigent (2002), if there are jumps in the asset price dynamics, then the multiple  $m$  must not be too high.

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