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Portfolio Management with Safety Criteria in Complete Financial Markets

Jean-Luc Prigent[a] and Salwa Toumi[b]
[a] *THEMA, University of Cergy, 33 bd du Port, 95011, Cergy, France*
Jean-luc.Prigent@eco.u-cergy.fr
[b] *ESSAIT, Tunis, Tunisie*
salwatoumi@hotmail.com

ABSTRACT

We examine portfolio asset management under safety constraints that control the probability that the portfolio return falls under a given reference level. We extend previous results of Roy (1952) and Kataoka (1963) that have been proved in a one-period setting to both multiperiod discrete-time and continuous-time models. Basic examples illustrate the results.

JEL: *C 61, G 11*

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I. INTRODUCTION

From the seminal work of Markowitz (1952), the mean-variance criteria are widely used by portfolio managers. Its generalization, based on the expected utility maximization of gains, has become also a standard criterion for portfolio optimization. In both cases, investors are supposed to prefer the more to the less and are risk averse. The mean-variance approach is justified under one of the following assumptions: (1) The asset returns are normally distributed and the investors' utilities are exponential; and (2) The utility functions are quadratic and the return distributions are characterized by their two first expectations. Moreover, as it has been proved in Markowitz and Levy (1979), the mean-variance approach is approximatively robust even when the assumptions (1) and (2) are not verified. For example, the quadratic approximations are often good local approximations of non-quadratic utility functions, when the asset returns distributions are not too asymmetric. Other criteria for portfolio choice are based on the geometric return, or the first four moments of returns (in particular, the skewness and the kurtozis).

In this paper, we examine portfolio optimization based on safety criteria: the investors focus on unfavourable events and want to limit the probability to get low portfolio returns. For example, the guaranteed funds managers deal with insured payments at maturity. They must provide minimum fixed returns to their clients while trying to obtain a better performance than a riskless return, by trading on financial markets. Therefore, they can introduce safety criteria such as Roy (1952) or Kataoka (1963) (see also Telser (1956)).

In the first section, we provide general results about optimal portfolio with respect to safety criteria in complete markets. The proofs are related to results of Föllmer and Leukert (1999) about option quantile hedging. In the second section, explicit solutions are given for basic examples such as the Cox-Ross-Rubinstein model and the Black and Scholes model. All proofs are gathered in the Appendix.

II. SAFETY CRITERIA WITHIN COMPLETE MARKETS

We analyze portfolio optimisation when the financial market is complete. Thus, in this framework, every contingent claim can be replicated by a self-financing financial strategy. Denote by R_p the return of a portfolio p and let R_{Min} be a fixed minimal portfolio return required by the investor. The Roy criterion consists in minimizing the probability that the return falls under the minimal return R_{Min}. Therefore, the Roy's optimization problem is to find the portfolio p that is the solution of:

$$\mathrm{Min}_p\, P(R_p < R_{Min}) \text{ or equivalently } \mathrm{Max}_p\, P(R_p \geq R_{Min})$$

The Kataoka criterion consists in choosing the portfolio with a "minimal return" R_{Min} being as high as possible under the following constraint limiting risk: the probability that the portfolio return is lower than R_{Min} must not exceed a given level ε. Consequently, the Kataoka's problem is:

$$\text{Max } R_{Min} \text{ and } P(R_p < R_{Min}) \leq \varepsilon.$$

In fact, Roy criterion focuses on the level of probability: for example, the investor may have some difficulty in estimating probability distributions if financial asset returns have rather large fluctuations. Therefore, he prefers to deal with a small level value ε. Kataoka's criterion is based on the choice of the "best" guaranteed return, for a given probability level. These two problems have been solved by Roy and Kataoka, in a one period setting, using also a normality assumption on asset returns (see Section Examples). In what follows, we show how these problems can be solved when the financial market is dynamically complete.

A. Roy Criterion and Dynamic Complete market

We assume that the asset price process is a semimartingale $X=(X_t)_{t\varepsilon[0,T]}$ defined on a probability space (Ω,F,P) with respect to a filtration $(F_t)_{t\varepsilon[0,T]}$. Recall that a self-financing strategy is defined by an initial capital $V_0 \geq 0$ and by an investment strategy ξ that is a predictable process. Such strategy (V_0, ξ) is called admissible if the process defined by $V_t=V_0+\int_{[0,t]} \xi_s \, dX_s$, t ε [0,T] is positive P-a.s. In a complete market, there exists a unique equivalent martingale measure that we denote by Q. According to Roy criterion, the investor has to solve the following problem: beginning with an initial capital V_0, he looks for the best admissible strategy that maximizes the probability that the portfolio terminal value verifies $V_T/V_0 \geq R_{Min}$. Therefore, he wants to maximize the probability of the event $V_T \geq R_{Min} V_0$. Consequently, the optimization problem is:

$$\text{Max}_{\{\xi\}} P[V_0+\int_{[0,t]} \xi_s \, dX_s \geq R_{Min} V_0] \quad (1)$$

1. Maximization of the success probability

For fixed R_{Min}, we call "success set" the set $A_{\{V0,\xi\}}$ defined by:

$$A_{\{V0,\xi\}} = \{V_T \geq R_{Min} V_0\}.$$

In a first step, we show that the Roy problem is equivalent to the determination of a success set of maximal probability. We denote by E_Q the expectation under the risk-neutral probability Q and by I_A the indicator function of the set A.

Proposition 1. Let A* be a solution of the problem:

$$\text{Max } P[A] \text{ A in } F_T \text{ and } E_Q[I_A] \leq 1/R_{Min}. \quad (2)$$

Let ς* be the perfect hedging strategy of the option $H=I_{A*}$. Then, $(V_0, \xi^*=V_0/E_Q[I_{A*}] \, \zeta^*)$ is the optimal solution of Problem (1).

The problem of constructing a maximal success set according to Roy criteria is then solved by applying Neyman-Pearson lemma, in a similar way as in the case of quantile theory in Föllmer and Leukert (1999). In fact, the Roy problem can be interpreted as quantile hedging the constant H= R_{Min} V_0 under the constraint that the initial capital V_0 to invest is smaller than H and that R_{Min} is greater than 1 (the riskless asset is taken as numeraire. Thus the riskless rate is assumed to be equal to 0). Since obviously $Q[A]=E_Q[I_A]$, Problem (2) is equivalent to maximise P[A] under the constraint $Q[A] \leq 1/R_{Min}$. This the reason why we can apply the Neyman-Pearson. For this, let denote by a* the threshold defined by:

$$A^*=\inf\{a: Q[dP/dQ > a\, H] \leq 1/R_{Min}\} \tag{3}$$

and consider the set A*={dP/dQ > a*}. We deduce:

Proposition 2. Consider the set A* defined by the two preceding relations. Then, if Q[A*]=a*, $(V_0, \xi^*=V_0/E_Q[I_{A^*}]\,\zeta^*)$ is the optimal solution of Problem (1).

2. Maximisation of the expected success ratio

As mentioned in Föllmer and Leukert (1999), the condition Q[A*]=a* is clearly satisfied when:

$$P[dP/dQ = a^*\, H] = 0 \tag{4}$$

Generally, it is not easy to find a set A* in F_T that satisfies the constraint (4). In this case, the Neyman-Pearson theory suggests to replace the "critical region" A with a random test, i.e, with a function φ, F_T-measurable such that $0 \leq \varphi \leq 1$. Let P be the set of these functions φ. Consider the following optimization problem:

$$E[\varphi^*]= \max_{\varphi \text{ in } P} E[\varphi], \text{ under the constraint } E_Q[\varphi^*] \leq 1/R_{Min} \tag{5}$$

The Neyman-Pearson lemma proves that the solution φ* of (5) is of the form:

$$\varphi^* = I\left\{\frac{dP}{Qd} > a^* H\right\} + \gamma I\left\{\frac{dP}{dQ} = a^* H\right\} \tag{6}$$

where a* is given by (3) and where γ is defined by (if the condition (4) is not verified):

$$\gamma = \frac{\frac{1}{R_{Min}} - Q\left[\frac{dP}{dQ} > a^* H\right]}{Q\left[\frac{dP}{dQ} = a^* H\right]} \tag{7}$$

This allows to give a solution to the extended Roy problem defined as follows: Let (V_0,ξ) be an admissible strategy. We define the "success ratio" associated to this strategy as

$$\varphi\{V_0,\zeta\} = I\{V_T \geq R_{Min} V_0\} + \frac{V_T}{R_{Min} V_0} I\{V_T < R_{Min} V_0\} \tag{8}$$

In fact, this new criterion is based on a mixed maximization of the success probability and the expectation of the ratio "Portfolio return/ minimum return" when the portfolio return is smaller than the minimum ratio R_{Min}.

In an extended version of the problem (1), we look for the strategy maximizing the expectation of the success ratio $E[\varphi_{\{V0,\xi\}}]$ under the probability P, over the set of all admissible strategies as in:

$$\text{Max}\{E[\varphi_{\{V0,\xi\}}]:(V_0,\xi) \text{ admissible}\} \tag{9}$$

Proposition 3. Let ζ be the process determining the perfect hedging of the F_T-measurable function φ^* defined by (6). Then the strategy $(V_0, (V_0/R_{Min})\,\zeta)$ is a solution to Problem (1).

B. Kataoka Criterion in Complete Market

To study this problem in a dynamic setting, we consider the same framework as in previous section (the study of Roy criteria). Following Kataoka criterion, the investor must look for a "minimal return" R_{Min} being maximal, to which he could associate an admissible strategy (V_0, ξ) such that the probability of the event $V_T \geq R_{Min} V_0$ is higher than some fixed level $1-\varepsilon$. Therefore, the problem is:

$$\text{Max } R_{Min,} \text{ under the constraint } P[V_0+\textstyle\int_{[0,t]} \xi_s\, dX_s \geq R_{Min} V_0] \geq 1-\varepsilon. \tag{10}$$

1. Maximization of the success probability

Let us optimize according to the Kataoka criterion, and define the Kataoka success set associated to an admissible strategy (V_0, ξ). In a first step, we reduce our problem (10) to constructing a maximal probability set. Note that obviously, searching for maximising R_{Min} is equivalent to minimize $1/R_{Min}$.

Proposition 4. Let A^* in F_T be a solution to the problem

$$\text{Min } E_Q[I_A], \text{ under the constraint } P[A] \geq 1-\varepsilon, \tag{11}$$

where Q is the unique equivalent martingale measure. Let $\alpha^*=1/\ E_Q[I_{A^*}]$ and let ζ^* be the perfect hedge of the option $H=\ I_{A^*}$. Then $R_{Min^*}=\alpha^*$ is the solution of the problem (11) for the strategy $(V_0,\ V_0\ R_{Min^*}\ \zeta^*)$.

Note that the problem (11) is equivalent to the maximization of $Q(A^c)$ under the constraint $P[A^c]<\varepsilon$. The solution is still given by Neyman-Pearson lemma as in the previous case. For this purpose, introduce a* defined by:

$$A^*=\inf\{a: P[dP/dQ > a] \leq \varepsilon\} \text{ and the set } A^{*c} =\{dP/dQ = a^*\} \quad (12)$$

Similarly to Proposition 2, we deduce:

Proposition 5. The set A^{*c} maximizes $Q[A^c]$ under the constraint $P[A^c] \geq \varepsilon$. Let ζ^* be the perfect replication of I_{A^*}. Then $(V_0,\ V_0\ \zeta^*,\ R^*_{Min}=a^*)$ is optimal for the problem (10).

Remark 1. Reminding that the Roy problem is equivalent to (2) and the Kataoka one is equivalent to (11), we can assert that there is a correspondence between the two problems. The behaviours of both strategies look similar as seen later on (see Section "Examples").

2. Maximization of the expected success ratio

Let (V_0,ξ) be an admissible strategy and, given $R_{Min} >1$. Define the "success ratio" associated to this strategy as

$$\varphi\{V_0,\zeta,R_{Min}\} = I\{V_T \geq R_{Min}V_0\} + \frac{V_T}{R_{Min}V_0} I\{V_T < R_{Min}V_0\} \quad (13)$$

Recall that P is the set of such functions φ. An extension of the original problem (10) consists in looking for strategies solution of the following optimisation problem:

$$\text{Min } E_Q\ [\varphi_{\{V0,\ \xi,\ RMin\ \}}], \text{ under the condition } E[\varphi_{\{V0,\ \xi,\ RMin\ \}}] \geq 1\text{-}\varepsilon \quad (14)$$

As we have done for the case of Roy criterion, we use Neyman-Pearson lemma to substitute the search for the success set for the search for a random test: a function φ F_T-measurable such that $0\leq \varphi \leq 1$. Then, we have to solve:

$$E[\varphi^*]= \min_{\varphi \text{ in } \square} E[\varphi], \text{ under the constraint } E[\varphi^*] \geq 1\text{-}\varepsilon \quad (15)$$

Due to the Neyman-Pearson lemma, the solution φ^* of Problem (15) is given by:

$$\varphi^* = I\left\{\frac{dP}{Qd} > a^*\right\} + \gamma I\left\{\frac{dP}{dQ} = a^*\right\} \quad (16)$$

where a* is given by (12) and where γ is defined by:

$$\gamma = \frac{(1-\varepsilon) - P\left[\frac{dQ}{dP} < a^*\right]}{P\left[\frac{dQ}{dP} = a^*\right]}, \tag{17}$$

if P[dQ/dP=a*] is not equal to 0. This allows to give a solution to this extended Kataoka problem:

Proposition 6. Let ξ^* be the perfect hedge of the F_T-measurable function φ^* solution of type (16) and let $V_0 = E_Q[\varphi^*]$. Then: (i) The success rate $\varphi_{\{V0,\ \xi,\ RMin\ \}}$ associated to (V_0, ξ^*,R_{Min}=1/V_0) is solution of Problem (14); and (ii) $\varphi_{\{V0,\ \xi,\ RMin\ \}} = \varphi^*$and $E[\varphi^*]=1-\varepsilon$.

Note that if A* defined by (12) verifies P[A*]=1-ε, then $\varphi^*=I_{A^*}$. Then, the solution of the Kataoka extended problem is also the solution of the Kataoka original problem. Note also that P[A*]=1-ε is verified if Q[dQ/dP=a*]=0.

III. EXAMPLES

A. Safety Criteria in A One-period Model

Assume that the return portfolio has a normal distribution (for example if the vector of asset returns is Gaussian) with mean R_p and variance σ_p. Then, after centering and reducing the variables, the distribution of R_p –$E[R_p]$ /σ_p is the standard Gaussian distribution N(0,1) and the problem is equivalent to solve :

$$\text{Max } P\left\{\frac{R_p - E[R_p]}{\sigma_p} < \frac{R_{Min} - E[R_p]}{\sigma_p}\right\} \tag{18}$$

Therefore, the optimization program is reduced to minimizing $\frac{E[R_p]-R_{Min}}{\sigma_p}$ or equivalently to maximizing $\frac{E[R_p]-R_{Min}}{\sigma_p}$. In that case, choosing a portfolio using Roy criteria consists in finding among all the lines starting from R_{Min} the one having the largest gradient. This largest feasible gradient corresponds to the tangent to the efficient frontier, starting from R_{Min}. The Kataoka criteria consists in choosing the portfolio with a "minimal return being maximal" under the following constraint limiting risk:

$$\text{Max } R_{Min} \quad \text{under } P\left[\frac{R_p - E[R_p]}{\sigma_p} < \frac{R_{Min} - E[R_p]}{\sigma_p}\right] \le \varepsilon \tag{19}$$

Thus, the gradient $\frac{E[R_p]-R_{Min}}{\sigma_p}$ must be equal to a given level which is the opposite of the ε-quantile of the distribution N(0,1). As an example, for ε=5%, the table of the distribution N(0,1) indicates that there is less than 5% of probability that the value of the Gaussian variable be smaller to (-1.65). Therefore, the previous condition is equivalent to $R_p \geq R_{Min}+1.65\ \sigma_p$. Note that, under the normality assumption, both Roy and Kataoka portfolios are necessarily mean-variance efficient. Besides, an investor choosing Roy portfolio instead of Kataoka one is more risk averse since he focuses on the risk probability level rather than on the return expectation. In this one-period setting, the portfolio payoff is obviously a linear combination of the available financial assets.

Remark 2. Assume that there exists only one risky asset S (for example, a financial index) and a riskless asset B (with rate r=0). Recall that we consider only the case $R_{Min} > R_B$ (with here R_B= 1).Then, the Roy problem is now to find the proportion θ to invest on the risky asset S such that Max $P[1+\theta\ (R_S-1) \geq R_{Min}]$.

If we impose the no short selling condition, then the optimal portfolio consists in setting θ=1. The investor chooses to invest his whole capital on the risky asset to maximize the probability to get a higher return than R_{Min}. Consequently, the optimal payoff V_T/V_0 is equal to S_T/S_0. Therefore, we have also Max $P[V_T/V_0 \geq R_{Min}]=P[S_T/S_0 \geq R_{Min}]$.

In what follows, we illustrate the effect of dynamical completeness. In particular, we prove that the optimal portfolio payoff is no longer an affine function of the underlying asset S.

B. Safety Criteria in the Cox-Ross-Rubinstein Framework

1. Application of Roy criteria to Cox-Ross-Rubinstein model

Given a price S_k (k=0,...,T) at date k, the price S_{k+1} at date k+1 can vary both by rising to the value S_k u with a probability π in (0,1) or by declining to the value S_k d where d<1<u. The probability space is then $\Omega=\{u,d\}^T$ and for each $\omega=(\omega_1,...,\omega_T\})$ in Ω, the price process is defined by $S_k(\omega)=S_0\prod_{j=1}^{k}\omega_j$, F_k being the filtration $\sigma(S_1,...,S_k)$ generated by the random variables $S_1,\ldots,S_k$. It is well known that in complete market, there exists an unique equivalent martingale measure Q under which the transition probability is given by:

$$Q[S_{k+1} = S_k\ u/F_k]=(1\text{-}d)/(u\text{-}d),\ Q[S_{k+1} = S_k\ d/F_k]=(u\text{-}1)/(u\text{-}d) \tag{20}$$

The density of Q in relation to P is given by :

$$dQ/dP=((1\text{-}d)/(\pi(u\text{-}d))^{N}\ ((u\text{-}1)/((1\text{-}\pi)(u\text{-}d)))^{T\text{-}N} \tag{21}$$

where $N(\omega)=\sum_{k=1}^{T} 1_{\{\omega k=u\}}$ follows the binomial law B(T,π). Note that $S_T=S_0\,u^N d^{T-N}$.
We have to determine $a^*=\inf\{a\text{: } P[dP/dQ > a] \leq 1/R_{Min}\}$.

We assume that T is big enough so we can obtain the approximation P[dP/dQ=a]=0. The Roy problem for fixed R_{Min} is then reduced to the determination of the set

$$A^*=\{dP/dQ > a^*\} \tag{22}$$

Let $c_1 = ((1-\pi)(u-d)/(u-1))^T$, $c_2 = S_0\, d^T$ and $R_1 = \pi(u-1)/(1-\pi)(1-d)$ and $R_2 = u/d$. Then the optimal set A* is such that

$$A^*=\{dP/dQ > a^*\}=\{ c_1R_1^{N} > a^*\}, \tag{23}$$

where a* is determined by the relationship $E_Q[I_{A^*}]=1/R_{Min}$. According to market parameters values, two cases must be examined:

Case 1: R_1>1. In that case, the risky asset has a positive trend, which is the usual assumption. Then, for T sufficiently high, the condition $c_1R_1^N > a^*$ is equivalent to the existence of an integer[1] $n_1=[[Ln(a^*/c_1)/Ln(R_1)]]+1\leq T$ such that the optimal set A* is characterized by $A^*=\{N \geq n_1\}=\{S_T \geq s_1 = c_2\, R_2^{n1}\}$.

The investor must hedge the option I_{A^*} which can be written as $I_{A^*} =I_{\{ST>s1\}}$. Therefore, for each R_{Min} value, there exists an integer $n_1(R_{Min})$ such that the success probability is given by

$$P(A^*R_{Min}) = \sum_{L=n1(R_{Min})}^{T} C_T^L \pi^L (1-\pi)^{T-L} \tag{24}$$

Case 2: R_1<1. For T big enough, there exists[(1)] $n_1=[[\, Ln(a^*/c_1)/Ln(R_1)\,]] \leq T$ such that $A^*=\{N \leq n_1\} = \{S_T \leq s_1 = c_2\, R_2^{n1}\}$. The investor must hedge the option $I_{\tilde{A}}$ which is written as $I_{A^*} = I_{\{ST<s1\}}$. The success probability is then given by:

$$P(A^*=\{N\leq n1\}) = P(A^*R_{Min}) = \sum_{L=0}^{n1(R_{Min})} C_T^L \pi^L (1-\pi)^{T-L} \tag{25}$$

In both cases, the strategy invested in the risky asset (the "Delta") is given by:

$$\theta_s^q(k) = \frac{E_Q[IA^*\,|\,S_{k+1}-S_k u]-E_Q[IA^*\,|\,S_{k+1}-S_k d]}{S_k(u-d)} \tag{26}$$

Numerical simulations. Let us fix the parameters: S_0=100, R_{Min}=1.2, B_0=1, T=250, π=0.5.

Case 1. We choose u and d such that $R_1 > 1$: for example u much higher than 1, or π sufficiently large such that the market has tendency to increase. This assumption is verified for the following standard approximation model: the probability π=1/2 and

$$u = 1+0.3/T+0.3/\sqrt{T} \text{ and } d = 1+0.3/T-0.3/\sqrt{T}.$$

According to Roy criteria, the optimal portfolio payoff is equal to a binary function. If the portfolio exceeds the minimal return R_{Min}, then the guarantee is satisfied. In fact, the objective of the agent is to maintain the final value of his portfolio to a value equal to $R_{Min}V_0$. In what follows, we illustrate numerically some features of Roy portfolio[2].

Case 1. $R_1>1$

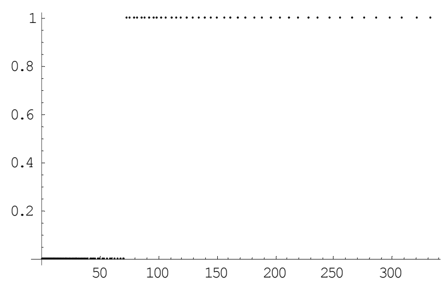

Figure 1: Roy payoff for R_{Min}=1.2

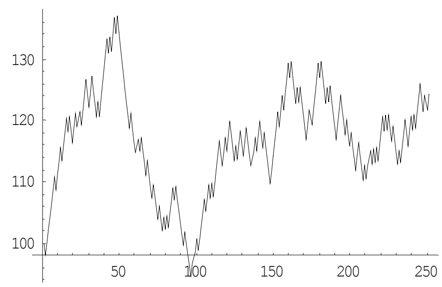

Figure 2: Random trajectory of S_t

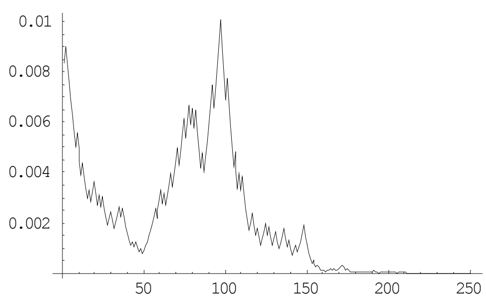

Figure 3: $\theta_{S(t)}$ for R_{Min} =1.2

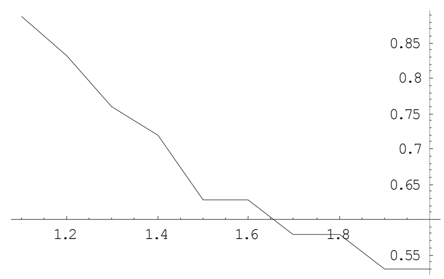

Figure 4: P success function of R_{Min}

Looking at previous curves, as mentioned above, the payoff of the Roy optimal portfolio is the payoff of a digital option: if the risky asset is very low, the investor has a null return. Otherwise, he gets exactly the return R_{Min} that he has chosen. It appears also that Roy Delta is a decreasing function of price S_t. If S_t increases, we need less risky assets to guarantee the required minimal return R_{Min}: $V_T \geq R_{Min}V_0$. Nevertheless, if S_t decreases we need to invest more. As regards the success probability, it is obvious that it diminishes if R_{Min} rises. We should note that this probability of success is significantly influenced by the variation of R_{Min} (variation between 0.9 to 0.55 for R_{Min} varying from 1 to 2).

Case 2. By choosing π, u and d such that $R_1<1$ ($<R_B$):

To have R_1<1, we must either choose d small or sufficiently diminish π to favour the jump to decline instead of the one to rise. The contingent claim price would have tendency to decline. For this purpose, we choose π=0.4 and keep the same values for the other parameters. As we have proved, in the case where $R_1 < 1$, the Roy option has an opposite appearance to the one in the first case. In the case $R_1 < 1$, as we have already noted, the market has tendency to decline. A manager chooses short sales of the risky asset and invests in the bond in order to increase the probability that $V_T \geq R_{Min}\ V_0$.

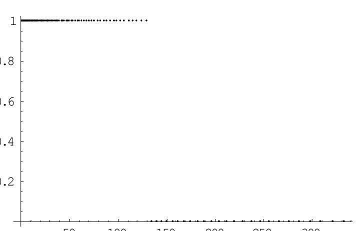

Figure 5: Roy payoff for R_{Min} =1.2

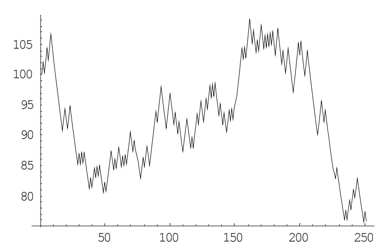

Figure 6: Random trajectory of S_t

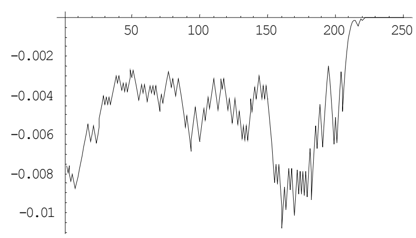

Figure 7: Delta of the return $\zeta_{S(t)}$ for R_{Min}=1.2.

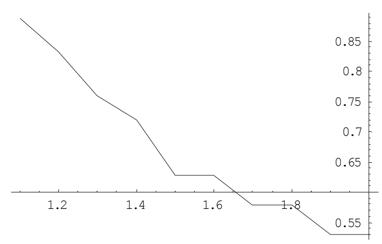

Figure 8: P success, function of R_{Min}

As mentioned above, the payoff of the Roy optimal portfolio is the payoff of a digital option. Now, if the risky asset is high, the investor has a null return. Otherwise, he gets exactly the return R_{Min} that he has chosen. This effect is due to the negative trend of the risky asset that induces high probability to get a low terminal value of S. Thus, the investor focuses on these events and adopts his strategy to get a "good" result for this "bad" case. Observing the fluctuations of the price, we note that when S_t decreases, the investor increases θ_{St} to reduce its short position (he buys the risky asset that he sold in short position with a higher price) and inversely if the price S_t increases, the investor sells more in short position since he expects price to decline. As regards the success probability given by $P[A=\{N \leq n_1\}]$, it is obvious that it diminishes if R_{Min} increases as in the first case (1). This decrease is influenced significantly by the variation of R_{Min}.

Remark 3. When comparing the Roy solutions (one-period model and dynamic complete model), we note that they differ appreciably. For the one-period model, under the assumption of no short-selling, the optimal return is simply equal to the risky asset S_T return. For the dynamic complete model, it is equal to 0 or R_{Min} according to the level of S_T.

2. Kataoka criterion and the Cox-Ross-Rubinstein model

We consider again the binomial model described in the Roy case. Then, we have to choose the hedging strategy of the option $I_{\tilde{A}}$ such that $A^*=\{dQ/dP < a^*\}$ and $P[A^*]=1-\varepsilon$.

Numerical simulations. We choose the same parameters as before:

Case 1. $R_l>1$:

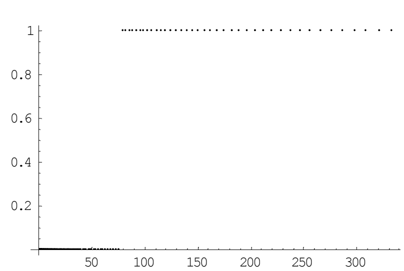

Figure 9: Kataoka payoff for ε=0.05

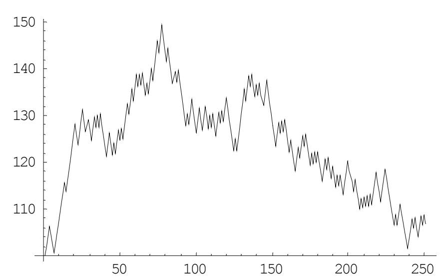

Figure 10: Random trajectory of S_t

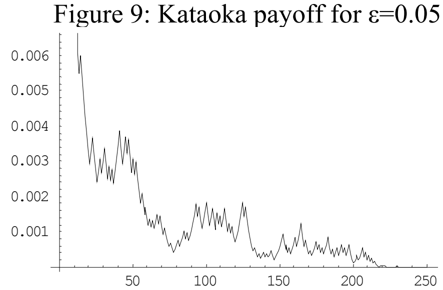

Figure 11: Delta of the return for ε=0.05.

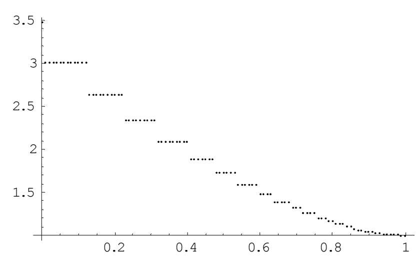

Figure 12: R_{Min} function of ε.

The binary option to replicate has the same form as in Roy case (with the same parameters values). The investor has to cover a binary option. Since $R_l>1$, he must invest less on S if the value of S_t increases and vice versa. For the last figure, we note that when the investor is more risk adverse ($\varepsilon\to 0$), the maximal R_{Min} obtained declines to one: $R_{Min}\to 1$. It means obviously that he prefers more and more to invest on the riskless asset.

Case 2. (By choosing π, u and d such that $R_1<1$):

As we have proved, Kataoka option in the case $R_1<1$ has an opposite appearance of the previous one (case $R_1>1$) and the same form as for the Roy model under the same parameters values.

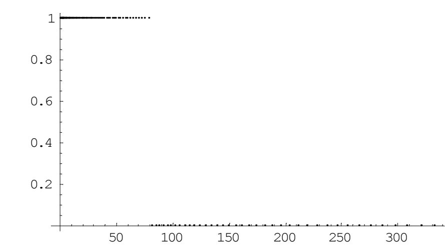

Figure 13: Kataoka payoff for $\varepsilon=0.05$.

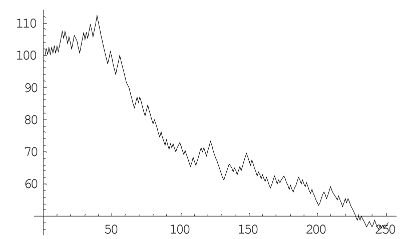

Figure 14: Random trajectory of S_t

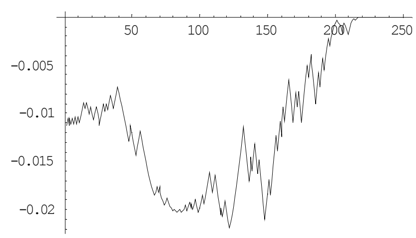

Figure 15: Delta of the return for $\varepsilon=0.05$.

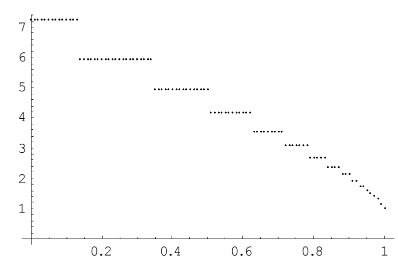

Figure 16: R_{Min} function of ε.

The Delta θ_{St} is still a decreasing function of the price S and $\theta_{St} < 0$. This is due to the market tendency to decline (since $R_1 < 1$). As for R_{Min}, it declines to be equal to one ($R_{Min}\to 1$) when ε converges to 0.

C. Safety Criteria in the Black and Scholes Framework

1. Roy criterion and Black and Scholes model

Recall that in the Black-Scholes model (i.e. for constant volatility $\sigma>0$), the price process of a risky asset is described by $dS_t=S_t\,(m\,dt+\sigma\,d\omega_t)$ where ω is a Brownian motion under P and m is constant. To simplify, the interest rate is assumed to be null r=0. The unique equivalent martingale measure is defined by:

$$\frac{dQ}{dP} = \exp\left[-\frac{m}{\sigma}\varpi T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right] \tag{27}$$

and since $\omega_t^* = \omega_t + (m/\sigma)\, t$ is a Brownian motion under Q, we have also:

$$S_T = S_0 \exp\left[\sigma \varpi_T^* - \frac{1}{2}\sigma^2 T\right] \tag{28}$$

Thus we deduce:

$$\frac{dQ}{dP} = \exp\left[\frac{m}{\sigma}\varpi T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right] = \beta (S_T)^{\frac{m}{\sigma^2}} \tag{29}$$

and

$$\beta = (1/S_0)^{(m/\sigma^2)} \exp[-(1/2)(m/\sigma^2)T + (m/2)T] \tag{30}$$

For fixed R_{Min}, the optimal Roy strategy is the replicating strategy of the option $H = I_{\tilde{A}}$, where A is written as $\tilde{A} = \{dQ/dP > \tilde{a}\}$ with $\tilde{a}$ determined by the relationship $E_Q[I_{\tilde{A}}] = 1/R_{Min}$. We have then $\tilde{A} = \{\beta\, (S_T)^{(m/(\sigma^2))} > a\}$.

Normally, we should distinguish two cases ($m > 0$ et $m < 0$). However, we consider only the first case $m > 0$ (market with price tendency to rise)[3]. In the case $m > 0$, the success set A can be written under the form $A = \{S_T > c\}$ where c is determined from the relationship $E_Q[I_{\tilde{A}}] = 1/R_{Min}$.

The Roy optimal portfolio has a payoff equal to this of an option that can be written as $H = I_{\tilde{A}} = I_{\{ST > c\}}$ and the Roy success probability is then $Q(\tilde{A}) = \Phi(-b/\sqrt{T})$ with b such that $c = S_0 \exp(\sigma b - (1/2)\sigma^2 T)$. Taking $d\_(c,t) = -(1/(\sigma\sqrt{T})) \operatorname{Ln}(c/S_t) - (1/2)\sigma\sqrt{(T-t)}$, the value V_t of the option to duplicate is equal to $V_t = E_{Qt}[I_{\tilde{A}}] = \Phi(d\_(c,t))$ and $b = -\sqrt{T}\Phi^{-1}(1/R_{Min})$. Besides, we determine the expressions of the Delta $\Delta(t,S_t) = (\partial V_t/\partial S_t)(t,S_t)$ and the Gamma $\Gamma(t,S_t) = (\partial^2 V_t/\partial S_t^2)(t,S_t)$ which allow us to study the variation of the quantity invested in the risky asset S. We obtain:

$$\begin{aligned} \theta_S^q(t) = \Delta(t,S_t) &= \frac{1}{\sqrt{2\pi}\sigma S_t} \exp\left[-d\_(c,t)^2/2\right] \\ \Gamma(t,S_t) &= \frac{1}{\sqrt{2\pi}\sigma S_t}\left[1 + \frac{1}{\sqrt{T}\sigma} d\_(c,t) \exp\left[-d\_(c,t)^2/2\right]\right] \end{aligned} \tag{31}$$

Numerical simulations. Let us observe the case where $R_{Min} = 1.2$.

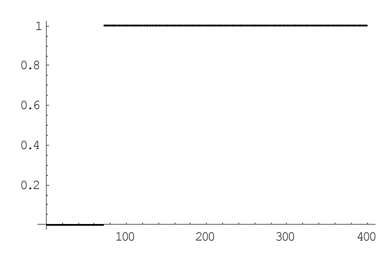

Figure 18: Roy payoff for R_{Min}=1.2

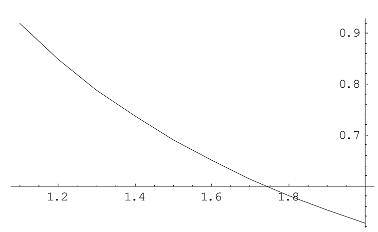

Figure 19: Prob. success function of R_{Min}

For large values of S_t, the investor chooses to reduce the quantity Δ (t,S_t) invested in the risky asset if S_t increases: Once the condition $V_T \geq R_{Min}V_0$ is achieved, the investor S_t is satisfied and the increase of the value of the asset does not interest him any more. However, for small values of S_t, $\Delta(t,S_t)$ is an increasing function of S_t.[4] The success probability is obviously a decreasing function of R_{Min} since it is more difficult to guarantee a larger return. Besides, $P[\tilde{A}]$ is an increasing function of m : larger is the drift term of the underlying asset, larger is the probability of success $P(V_T \geq R_{Min}V_0)$.
The probability $P[\tilde{A}]$ is also a decreasing function of the volatility σ: Actually, the event $V_T \geq R_{Min}V_0$ is less probable at the expiry date if the volatility is larger.

2. Kataoka criterion and Black and Scholes model

Using same approach, we deduce[5]

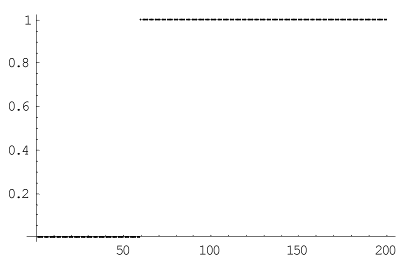

Figure 18: Kataoka payoff for R_{Min}=1.2

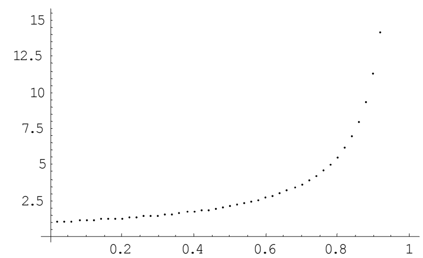

Figure 19: R_{Min} function of Prob. Success

The investor invests more in the risky asset if S_t decreases and vice versa in order to respect the guarantee $P(V_T \geq R_{Min}V_0) \geq 1\text{-}\varepsilon$. For high values of ε (near 1), he wants only to maximize the minimum return R_{Min}. This is the reason why we observe that the curve $R_{Min}(\varepsilon)$ is strongly increasing for values of S near 1. Conversely, when ε is small (near 0), the investor is absolutely averse towards "probability risk". But this implies that the minimum guarantee return R_{Min} is smaller (near 1).

D. Safety criteria with several risky assets

In this section, we extend previous results to the multi asset case. We assume that there exist n risky assets S_i (usually stocks) and a n-dimensional Brownian motion $(W_t)_t$ such that, for all i:

$$dS_{i,t}=S_{i,t}[m_i\, dt+\textstyle\sum_j \sigma_{i,j}\, dW_{j,t}], \tag{32}$$

where parameters the vector m= (m_i) and $\sigma_{i,j}$ are constant and the volatility matrix $\Sigma=[\sigma_{i,j}]_{i,j}$ is invertible. Note that we obtain $W_T=\Sigma^{-1}X+A$, where X is the vector $X_i=\ln(S_{i,T}/S_{i,0})$ and A = T[m-(1/2)B] with $B_i = \sum_j \sigma_{i,j}^2$. Then, there exists an unique risk-probability Q determined from dQ/dP =E [Λ.W], where E denotes the Dade-Doléans exponential and the vector Λ is determined from the Girsanov condition $\Sigma\Lambda = m$. Then, the optimal Roy portfolio has still the payoff of a binary option with the following form:

$$I_{\{\Pi_i S_i^{\alpha} > d\}} = I_{\{\phi . X > e\}}\ , \tag{33}$$

where Φ is fixed vector depending only on market parameters and on the minimal return R_{Min} through the condition $P[\Phi.X > d] = 1/R_{Min}$. Similar result can be deduced for the Kataoka criterion.

IV. CONCLUSION

We have established the characterization of portfolio strategies according to Roy and Kataoka criteria, in complete markets. The investor is assumed to maximize the probability that his portfolio return is higher than a given minimal return. To illustrate main properties, first we have examined complete markets with one risky asset, both in binomial model and Black and Scholes settings. We have shown that the investor acts as if he should hedge a binary option, depending on the tendency of the underlying asset to rise or to decline. The investor disinvests in the underlying asset if the market has tendency to rise and invests if the market has tendancy to fall. This is explained by the fact that the main investor's objective is to maintain the portfolio return above R_{Min}. However, he may suffer severe losses when his portfolio return is smaller than the minimal return chosen. Such result can be examined for incomplete markets as for example in Prigent and Toumi (2002). However explicit solutions are generally not available in the incomplete framework.

ACKNOWLEDGEMENTS

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ENDNOTES

1. The symbol [[ ]] refers to the integer part.
2. Note that obviously for a given number of dates T, the asset prices only have a finite number of values. Thus, the curves are only composed of dots.
3. We illustrate only the case $m > 0$ since it is similar to the binomial case with $R_1 > 1$. The case $m < 0$ would be similar to the case $R_1 < 1$ (market with tendency to decline).
4. Note that to draw the success probability curve by varying the parameters R_{Min}, m and σ, we use the relation $P(A^*) = \Phi(-(b-(m/\sigma)T)/(\sqrt{T}))$.

5. See Prigent and Toumi (2002) or Toumi (2003) for more details about these results.

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APPENDIX

Proof of Proposition 1.

I) Let (V_0,ξ) be an admissible strategy. The process V_t defined by is a positive local martingale under Q. From the definition of $A_{\{V0,\xi\}}$, we deduce that $V_T/(V_0 R_{Min})\geq I_{A\{V0,\xi\}}$ P-a.s. Thus, since Q is equivalent to P, we have:

$$E_Q[V_T/(V_0 R_{Min})] \geq E_Q[I_{A\{V0,\xi\}}]. \qquad (34)$$

Therefore, since $E_Q[V_T]=V_0$, we obtain $E_Q[I_{A\{V0,\xi\}}]\leq 1/R_{Min}$, and consequently $A_{\{V0,\xi\}}$ satisfies the constraint (2) for any admissible strategy. Then, we deduce:

$$P[A_{\{V0,\xi\}}] \leq P[\tilde{A}]. \qquad (35)$$

II) Let ζ be the perfect hedging of the contingent claim $I_{\tilde{A}}$. Consider the portfolio with initial value V_0 and return equal to $\alpha\, I_{\tilde{A}}$ (therefore, $V_T=V_0\,\alpha\, I_{\tilde{A}}$). Then, we need to prove that $(V_0,\xi=\alpha\,\zeta)$ is an optimal solution for the problem (1). For this, note that

$$P[V_T/V_0\geq R_{Min}] = P[\alpha\, I\_\{A\}\geq R_{Min}] = P[I\_\{A\}\geq R_{Min}/\alpha]. \qquad (36)$$

Moreover, since $R_{Min}/\alpha \leq 1$,$P[I_{\tilde{A}} \geq R_{Min}/\alpha] \leq P[I_{\tilde{A}} \geq 1] = P[A]$. Therefore, $P[V_T/V_0\geq R_{Min}] \geq P[A]$. Since $\tilde{A}$ is the solution of Problem (2), we have also $P[V_T/V_0\geq R_{Min}] \leq P[A]$. Consequently, we deduce the equality, which proves the result.

Proof of Proposition 2.

According to Neyman-Pearson, $\tilde{A}$ is a solution to (2) (see Föllmer and Leukert (1999) for more details) and then, from preceding proposition, $(V_0,\xi=V_0\,\alpha\,\zeta)$- where ζ is the perfect hedging of the option $I_{\tilde{A}}$- is the optimal solution of Problem (1).

Proof of Proposition 3

The argument is similar to the proof of the preceding proposition. Note that if the condition (4) is verified, then this implies that $\varphi^* = I_{\tilde{A}}$ (due to the definition of φ^*). In this case, ζ is the perfect hedging of $I_{\tilde{A}}$ and $E_Q[I_{\tilde{A}}]=1/R_{Min}$. Then, the strategy $(V_0,V_0/R_{Min}\,\zeta)$ is the solution of the extended version of the Roy problem (9) and also the solution to the original problem (1).

Other proofs are detailed in Prigent and Toumi (2002) and Toumi (2003).